Recall :

Thm. 4.12 Let V be a vectorspace and $B = \{b_1, \dots, b_n\}$ a basis for V. Then for each $v \in V$ there exist unique scalars $\lambda_1, \dots \lambda_n \in \mathbb{R}$ such that :

$$v = \lambda_1 b_1 + \dots + \lambda_n b_n$$

Def. We call the vector
$$[v]_B := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$
the coordinate vector of v in the basis B.

Ex. A $V = \mathbb{R}^2$, $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, $w = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Q: $[w]_B = ?$

Answer : $w = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies [w]_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Q : Is there a method that helps us to quickly find the representation of a vector in a desired basis ?

Q : If we already have the representation of a vector in some basis, can we just "translate" to a different basis ?

→ We will start answering these questions for the case where $V = \mathbb{R}^n$.

Let us go back to Example A:

$w = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ expressed in the basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is $[w]_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Because: $1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

This can be rewritten as: $\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{=M} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Let M be the matrix whose columns are the vectors in B (in order!) then: $M [w]_B = w$

Since M is invertible, we can also write this as:

$$M^{-1} w = [w]_B$$

Coordinates and coordinate change in $\mathbb{R}^n$

Def. Let $B = \{b_1, \dots, b_n\}$ be an ordered basis of $\mathbb{R}^n$.

We call the matrix $P_B := (b_1, \dots, b_n) \in \mathbb{R}^{n \times n}$ the change of coordinate matrix from B to the standard basis.

Reason: If $x \in \mathbb{R}^n$ then $x = \underbrace{P_{\mathcal{B}} \cdot [x]_{\mathcal{B}}}_{= x_1 b_1 + \dots + x_n b_n}$

So multiplying $P_{\mathcal{B}}$ to a vector written in basis $\mathcal{B}$, yields the vector written in the standard basis.

Since $\mathcal{B}$ is a basis, its vectors are independent. Thus, $\det P_{\mathcal{B}} \neq 0$ and $P_{\mathcal{B}}$ is invertible.

$$\Rightarrow \quad P_{\mathcal{B}}^{-1} x = [x]_{\mathcal{B}} \quad \forall x \in \mathbb{R}^n$$

Def. We call $P_{\mathcal{B}}^{-1}$ the change of coordinate matrix from the standard basis to $\mathcal{B}$.

We can also switch between two bases $\mathcal{B}$ and $\tilde{\mathcal{B}}$ of $\mathbb{R}^n$, neither of which is the standard basis:

For $x \in \mathbb{R}^n$:

$$\underbrace{\left(P_{\tilde{\mathcal{B}}}^{-1} \cdot P_{\mathcal{B}}\right)}_{=: P_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}}} [x]_{\mathcal{B}} = P_{\tilde{\mathcal{B}}}^{-1} \left(\underbrace{P_{\mathcal{B}} [x]_{\mathcal{B}}}_{= x}\right) = P_{\tilde{\mathcal{B}}}^{-1} x = [x]_{\tilde{\mathcal{B}}}$$

Def. $P_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}}$ is the change of coordinate matrix from $\mathcal{B}$ to $\tilde{\mathcal{B}}$.

Ex. $\mathcal{B} = \left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\4\\9\end{pmatrix}, \begin{pmatrix}1\\0\\-3\end{pmatrix} \right\}$, $\tilde{\mathcal{B}} = \left\{ \begin{pmatrix}0\\0\\3\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$

are bases of $\mathbb{R}^3$. Let $w \in \mathbb{R}^3$ so that $[w]_{\mathcal{B}} = \begin{pmatrix}1\\0\\1\end{pmatrix}$.

Find $[w]_{\tilde{\mathcal{B}}}$.

Solution:

$$P_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}} = P_{\tilde{\mathcal{B}}}^{-1} P_{\mathcal{B}} = \underbrace{\begin{pmatrix}0&0&1\\0&2&0\\3&0&0\end{pmatrix}^{-1}}_{= \begin{pmatrix}0&0&1/3\\0&1/2&0\\1&0&0\end{pmatrix}} \begin{pmatrix}1&4&1\\2&4&0\\3&9&-3\end{pmatrix} = \begin{pmatrix}1&3&-1\\1&2&0\\1&4&1\end{pmatrix}$$

$$\Rightarrow [w]_{\tilde{\mathcal{B}}} = P_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}} [w]_{\mathcal{B}} = \begin{pmatrix}1&3&-1\\1&2&0\\1&4&1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}$$

The coordinate mapping

Thm. 4.13 Let V be a vector space and $\mathcal{B} = \{b_1, \dots, b_n\}$ a basis for V. Define the mapping $T: V \to \mathbb{R}^n$ by:

$$T(v) := [v]_{\mathcal{B}}$$

This map T is linear, bijective and its inverse is linear as well.

Notice: In the case when $V = \mathbb{R}^n$, we already know this to be true as then $T(v) = P_{\mathcal{B}}^{-1} \cdot v$.

In the case where $V \neq \mathbb{R}^n$ (but e.g. V is a polynomial space) Thm. 4.12 allows us to from now on identify that space with $\mathbb{R}^n$.

e.g. $\mathbb{P}_2$ with basis $\mathcal{B} = \{1, x, x^2\}$. $\Rightarrow n = 3$

and we identify $\mathbb{P}_2$ with $\mathbb{R}^3$ by:

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

So we can think of polyn. as actual vectors!

This will have many useful applications!

So let us prove the Thm.

<u>Proof</u> 1.) Linearity We have to show:

$$[x + \lambda y]_{\mathcal{B}} = [x]_{\mathcal{B}} + \lambda [y]_{\mathcal{B}} \quad \forall x, y \in V, \lambda \in \mathbb{R}$$

Let $x = \alpha_1 b_1 + .. + \alpha_n b_n$ and $y = \beta_1 b_1 + .. + \beta_n b_n$

$$\Rightarrow \quad x + \lambda y = (\alpha_1 b_1 + .. + \alpha_n b_n) + \lambda(\beta_1 b_1 + .. + \beta_n b_n)$$

$$= (\alpha_1 + \lambda\beta_1) b_1 + .. + (\alpha_n + \lambda\beta_n) b_n$$

Calculus rules in V

$$\Rightarrow [x]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad [y]_{\mathcal{B}} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \quad [x + \lambda y]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 + \lambda\beta_1 \\ \vdots \\ \alpha_n + \lambda\beta_n \end{pmatrix}$$

Hence by $\mathbb{R}^n$ calculus: $[x + \lambda y]_{\mathcal{B}} = [x]_{\mathcal{B}} + \lambda [y]_{\mathcal{B}}$

2.) Injectivity We have to show: If $[v]_B = 0$ then $v = 0$

$$\text{If } [v]_B = 0 \implies v = \underbrace{0 \cdot b_1}_{=0_V} + \dots + \underbrace{0 \cdot b_n}_{=0_V} \overset{\uparrow}{=} 0_V.$$

calculus in V

3.) Surjectivity We have to show: For every $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$
we have to find $v \in V$ so that $[v]_B = x$.

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$. Define $v = x_1 b_1 + \dots + x_n b_n \implies [v]_B = x$.

4.) Linearity of T^{-1}

<u>Claim</u>: If a linear map $T: V \to W$ is invertible,
then $T^{-1}: W \to V$ is also linear.

<u>Proof</u>: Let $w_1, w_2 \in W$, $\lambda \in \mathbb{R}$.

To show: $T^{-1}(w_1 + \lambda w_2) = T^{-1}(w_1) + \lambda T^{-1}(w_2)$.

Define $v_1 := T^{-1}(w_1)$, $v_2 := T^{-1}(w_2)$
$\implies w_1 = T(v_1)$, $w_2 = T(v_2)$

Hence: $T^{-1}(w_1 + \lambda w_2) = T^{-1}(T(v_1) + \lambda T(v_2))$

$$\underset{T \text{ linear}}{=} T^{-1}(T(v_1 + \lambda v_2)) \underset{T^{-1}T = \mathrm{id}}{=} v_1 + \lambda v_2$$

$$= T^{-1}(w_1) + \lambda T^{-1}(w_2).$$

$\square$

Thm. 4.14 Let V, W be vector spaces and $T: V \to W$ an invertible linear transformation. Then:

(i) If $v_1, \dots, v_k \in V$ are linearly indep.

$\Leftrightarrow T(v_1), \dots, T(v_k)$ are linearly indep.

(ii) If $\text{Span}\{v_1, \dots, v_k\} = V$ $\Rightarrow$

$\Leftrightarrow \text{Span}\{T(v_1), \dots, T(v_k)\} = W$

(iii) If $B = \{v_1 \dots v_k\}$ is a basis of V

$\Leftrightarrow T(B) := \{T(v_1), \dots, T(v_k)\}$ is a basis of W

Proof (i) Assume that $v_1, \dots, v_k \in V$ are linearly indep.

Let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and consider $\lambda_1 T(v_1) + \dots + \lambda_k T(v_k) = 0$

Since T is linear:

$$0 = \lambda_1 T(v_1) + \dots + \lambda_k T(v_k) = T(\lambda_1 v_1 + \dots + \lambda_k v_k)$$

Since T is invertible:

$$0 = T^{-1}(0) = \lambda_1 v_1 + \dots + \lambda_k v_k$$

Since $v_1, \dots, v_k \in V$ are linearly indep.

$\Rightarrow \lambda_1 = \dots = \lambda_k.$

(ii) Assume that $\mathrm{Span}\{v_1, \dots, v_k\} = V$.

Let $w \in W$. To show: $\exists \lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that

$$\lambda_1 T(v_1) + \dots + \lambda_k T(v_k) = w$$

T invertible. So there exists $v := T^{-1}(w) \in V$

Since $\mathrm{Span}\{v_1, \dots, v_k\} = V$.

$\exists \lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that $\lambda_1 v_1 + \dots + \lambda_k v_k = v$

T linear, so: $$w = T(v) = T(\lambda_1 v_1 + \dots + \lambda_k v_k)$$
$$= \lambda_1 T(v_1) + \dots + \lambda_k T(v_k)$$

(iii) Follows from (i) and (ii). $\square$

<u>Cor. 4.15</u> Let V be a vector space and $B = \{b_1, \dots, b_n\}$ a basis. Let $S = \{v_1, \dots, v_k\} \subset V$ be a set of vectors and let $\tilde{S} = \{[v_1]_B, \dots, [v_k]_B\} \subset \mathbb{R}^n$ be the coordinate vectors of the elements of S for the basis B. Then:

(a) S is lin. indep. (in V) if and only if $\tilde{S}$ is lin. indep. (in $\mathbb{R}^n$)

(b.) $\mathrm{Span}(S) = V$ if and only if $\mathrm{Span}(\tilde{S}) = \mathbb{R}^n$.

(c.) S is a basis for V if and only if $\tilde{S}$ is a basis for $\mathbb{R}^n$.

<u>Proof</u>: Combine Thms 4.13 and 4.14. $(W = \mathbb{R}^n, T(v) := [v]_B)$

Thms 4.13 and 4.14 as well as Cor. 4.15 allow us to identify every vector space V that has a finite basis $\{b_1, \dots, b_k\}$ with with the vector space $\mathbb{R}^k$. The elements of b_i of the basis are identified with the elements e_i of the standard basis of $\mathbb{R}^k$.

Here are two useful applications of this conclusion:

<u>Ex.</u> Consider $V = \mathbb{P}_2$.

We know that $B = \{1, x, x^2\}$ is a basis of $\mathbb{P}_2$

So the transformation $T: \mathbb{P}^2 \to \mathbb{R}^3$, $T(p) := [p]_B$

identifies $\mathbb{P}^2$ with $\mathbb{R}^3$. (Thm. 4.13)

In particular 1 is identified with $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, x with $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

and x^2 with $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Now consider the set $S = \{ 1+2x^2, 4+x+5x^2, 3+2x \} \subset \mathbb{P}_2$

<u>Q</u>: Is S a basis for $\mathbb{P}_2$?

From Cor. 4.15, we know:

S is a basis of $\mathbb{P}_2 \iff T(S)$ is a basis of $\mathbb{R}^3$

So let's compute $T(S)$:

$$T(\underline{1+2x^2}) = [1+2x^2]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$T(\underline{4+x+5x^2}) = [4+x+5x^2]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$$

$$T(\underline{3+2x}) = [3+2x]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

So $T(S) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right\}$. Is it a basis of $\mathbb{R}^3$?

$\det \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{pmatrix} = \ldots$ compute $\ldots = 0 \implies T(S)$ is not a basis of $\mathbb{R}^3$

$\implies S$ is not a basis of $\mathbb{P}_2$.

<u>Ex.</u> Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be a linear transformation

defined by $T(p) = \begin{pmatrix} p(0) \\ p'(1) \end{pmatrix}$

Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$.

Identify $\mathbb{P}_2$ with $\mathbb{R}^3$: $a_0 + a_1 x + a_2 x^2 \mapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$ (as above)

Let $\tilde{T}: \mathbb{R}^3 \to \mathbb{R}^2$ be the map $T: \mathbb{P}_2 \to \mathbb{R}^2$ after identification

(or "translation") of $\mathbb{P}_2$ into $\mathbb{R}^3$.

i.e we want $\tilde{T}$ to be "the same as T" up to the translation $V \leftrightarrow \mathbb{R}^3$.

i.e. for every $p = a_0 + a_1 x + a_2 x^2$ we want

$$T(p) = T\underbrace{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}}_{\text{the "translated } p\text{"}}$$

$$\begin{tikzcd} \mathbb{P}^2 \arrow[dr, "T"] \arrow[d, leftrightarrow, "\text{translate}"'] & \\ \mathbb{R}^3 \arrow[r, "\tilde{T}"'] & \mathbb{R}^2 \end{tikzcd}$$

Find the matrix M of the linear map $\tilde{T}: \mathbb{R}^3 \to \mathbb{R}^2$.

(Next week, we will learn how to do this more systematically for more difficult examples.)

$$\Rightarrow \quad M \in \mathbb{R}^{2\times 3} \quad \text{so that} \quad M \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = T(a_0 + a_1 x + a_2 x^2) \in \mathbb{R}^2$$

$$T(\underbrace{a_0 + a_1 x + a_2 x^2}_{=p}) = \begin{pmatrix} p(0) \\ p'(1) \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 + 2a_2 \end{pmatrix}$$

$$\Rightarrow \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{because} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 + 2a_2 \end{pmatrix}$$

Find basis of $\mathrm{Ker}(M)$ and $\mathrm{Ran}(M)$ as taught last week:

$$\Rightarrow \quad B_{\mathrm{Ker}(M)} = \left\{ \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\} \subset \mathbb{R}^3, \quad B_{\mathrm{Ran}(M)} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\Rightarrow \quad B_{\mathrm{Ker}(T)} = \{ 2x - x^2 \} \qquad B_{\mathrm{Ran}(T)} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Cor 4.15

4.5 The dimension of a vector space

Cor. 4.15 has prepared us to finally define the dimension of a vector space.

__Thm. 4.16__ Let V be a vector space and $\mathcal{B}, \tilde{\mathcal{B}}$ two bases for V. Then $\mathcal{B}$ and $\tilde{\mathcal{B}}$ have the same number of elements.

This means: If $\#\mathcal{B} = n \Rightarrow \#\tilde{\mathcal{B}} = n$

If $\#\mathcal{B} = \infty \Rightarrow \#\tilde{\mathcal{B}} = \infty$

(In this course we do not distinguish different types of infinite.)

__Proof__ First case: $V = \mathbb{R}^n$. We know $\{e_1 \dots e_n\}$ is a basis.

Assume $\mathcal{B} = \{b_1 \dots b_m\}$ is another basis. To show: $n = m$.

$$\left.\begin{array}{l} \text{If } m > n, \text{ then } \mathcal{B} \text{ is dependent (by Thm 1.8)} \\ \text{If } n > m, \text{ then } \operatorname{Span}(\mathcal{B}) \neq \mathbb{R}^n \text{ (by Thm 1.14 + 1.15)} \end{array}\right\} \underset{\mathcal{B} \text{ is a basis}}{\Rightarrow} n = m.$$

__Second case:__ $V \neq \mathbb{R}^n$ but has a finite basis $\mathcal{B} = \{b_1, \dots b_n\}$.

Assume $\tilde{\mathcal{B}} = \{\tilde{b}_1 \dots \tilde{b}_m\}$ is another basis.

To show: $n = m$.

Since $\mathcal{B}$ is a basis of V, the coordinate map

$$T: V \to \mathbb{R}^n, \quad T(v) := [v]_{\mathcal{B}} \in \mathbb{R}^n$$

is a linear bijection (by Thm. 4.13)

Now consider $T(\tilde{\mathcal{B}})$:

Write $\tilde{\mathcal{B}} = \{\tilde{b}_1, \dots, \tilde{b}_m\}$, then: $T(\tilde{\mathcal{B}}) = \{T(\tilde{b}_1), \dots, T(\tilde{b}_m)\}$.

By Cor. 4.15, since $\tilde{\mathcal{B}}$ is a basis of V, $T(\tilde{\mathcal{B}})$ is a basis of $\mathbb{R}^n$.

So by First Case: $m = n$.

<u>Last case</u>: V has an infinite basis $\mathcal{B}$.

Let $\tilde{\mathcal{B}}$ be another basis.

To show: $\tilde{\mathcal{B}}$ is also infinite.

Assume for a contradiction that $\tilde{\mathcal{B}}$ is finite

$\Rightarrow$ $\tilde{\mathcal{B}} = \{\tilde{b}_1, \dots, \tilde{b}_n\}$

$\Rightarrow$ we can identify V with $\mathbb{R}^n$ using $T(v) := [v]_{\tilde{\mathcal{B}}}$ (Thm 4.13)

$\Rightarrow$ $\mathcal{B}$ also consists of n elements

as in second case

$\Rightarrow$ Contradiction. $\square$

Def. Let V be a vector space.

- If V has a finite basis $\mathcal{B}$ consisting of $n \in \mathbb{N}$ elements, then we call V a finite dimensional vector space and we call n the dimension of V : $\dim V = n$.

- If V has a basis $\mathcal{B}$ consisting of infinitely many elements, we call V an infinite dimensional vector space and we can write $\dim V = \infty$

- If $V = \{0\}$ then $\dim V := 0$.

  has no basis!

Ex.
- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathbb{R}^{m \times n}) = m \cdot n$
- $\dim(\mathbb{P}_n) = n+1$
- $\dim(\mathbb{P}) = \infty$
- $\dim\left(\operatorname{Span}\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}\right\}\right) = 2$
- $\dim\left(\underbrace{\operatorname{Span}\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix}\right\}}_{= \operatorname{Span}\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}\right\}}\right) = 1$

Subspaces of finite dimensional spaces

Thm 4.17 (The basis expansion theorem)

Let V be a vector space with $\dim V < \infty$ and $H \subseteq V$ a subspace. Then:

(1) $\dim H \leq \dim V$

(2) every linearly independent set $S \subseteq H$ can be expanded to a basis of H.

By (2) we mean: If $S = \{s_1, \dots, s_k\} \subseteq H$ indep. then either it is a basis or we can find $s_{k+1} \dots s_m \in H$ so that $B := \{s_1 \dots, s_k, s_{k+1} \dots s_m\}$ is a basis of H.

Ex. $\mathbb{P}_2 \subseteq \mathbb{P}_3$, $\dim(\mathbb{P}_3) = 4 < \infty$

$S = \{1+x, 3x\}$ is indep. but not a basis.

Can we add some other element(s) of $\mathbb{P}_2$ to turn S into a basis of $\mathbb{P}_2$?

→ we could add x^2: $B = \{1+x, 3x, x^2\}$ is a basis of $\mathbb{P}_2$.

or x^2+1
or $x - 5x^2$
or ...

This only works for subspaces H of a finite-dimensional V

<u>Ex.</u> $V = \mathbb{P}$ $(\dim V = \infty)$

$H := \{ p \in \mathbb{P} : a_i \in \mathbb{Z} \} \Rightarrow H \subset V$ subspace

$S = \{1, x, x^2\}$ independent but not a basis

<u>Q</u>: Can we expand S to a basis of H by adding (finitely many) elements of H to S? $\rightarrow$ No!

<u>Proof of Thm 4.17</u> Let $\dim V = n$.

First case: $H = \{0_V\}$.

Then $\dim H = 0$ by def. and $0 < n$. So (1) is satisfied.

Moreover, H does not contain any indep. subset.

So there is nothing to check regarding (2).

Second case: $H \neq \{0_V\}$. Let $S = \{s_1, \dots, s_k\}$ be a lin. indep. subset.

- If $\mathrm{Span}(S) = H$ then S is a basis of H $\rightarrow$ (2) ✓

  And $\dim H = \# S = k$.

  Moreover, since S indep. (in H and hence in V):

  by Thms 1.8 + 4.14 it follows that $k \leq n \Rightarrow$ (1) ✓

- If $\mathrm{Span}(S) \neq H$ we claim that:

$\exists\, v \in H$ such that $\{s_1, \dots, s_k, v\}$ indep.

and we will call this $v =: s_{k+1}$

If the claim is true, this means that we can just keep adding new s_i to S until one of the two scenarios occurs:

(A) The span is equal to H

(B) The set contains n elements.

If (B) occurs, then the the expanded set S (let us call it $\tilde{S}$) contains n indep. elements.

Hence it is a basis of V. So $\mathrm{Span}\, \tilde{S} = V$

$$\left.\begin{array}{l} \text{Since } \tilde{S} \subseteq H \Rightarrow \mathrm{Span}\, \tilde{S} \subseteq H \\ \text{And } H \subseteq V = \mathrm{Span}\, \tilde{S} \end{array}\right] \Rightarrow H = \mathrm{Span}\, \tilde{S} \qquad (2) \checkmark$$

And $H = V \Rightarrow \dim H = \dim V$. (1) ✓

If (A) occurs, then $\mathrm{Span}(\tilde{S}) = H \Rightarrow (2)$ ✓

and by Thm. 4.15 $\# \tilde{S} < n$.

So $\dim H = \# \tilde{S} < n = \dim V \Rightarrow (1)$ ✓

We are left to prove the claim:

Assume Span $S \neq H$. Also we know S indep.

Let $h \in H$ such that $h \notin \mathrm{Span}(S)$ and understand independence / dependence of $\{s_1, \dots, s_k, v\}$:

What are solutions of the equation

$$(*) \quad \lambda_1 s_1 + \dots + \lambda_n s_n + \lambda_{k+1} v = 0 \quad ?$$

If $\lambda_{k+1} = 0$ then $(*)$ is:

$$\lambda_1 s_1 + \dots + \lambda_n s_n + 0 = 0$$

So by independence of S $\Rightarrow \lambda_1 = \dots = \lambda_k = 0 = \lambda_{k+1}$

If $\lambda_{k+1} \neq 0$ then $(*)$ can be written as:

$$v = \frac{1}{\lambda_{k+1}} (\lambda_1 s_1 + \dots + \lambda_n s_n)$$

$\Rightarrow v \in \mathrm{Span}\, H$ ↯

In conclusion: $\lambda_1 = \dots = \lambda_{k+1} = 0$ is the only solution for $(*)$.

Hence $\{s_1 \dots s_k, v\}$ is indep. $\square$

Thm 4.18 Let V be a vector space with $\dim V = n < \infty$ and $\mathcal{B} = \{b_1, \dots, b_n\} \subseteq V$. Then the following are equivalent:

(1) $\mathcal{B}$ is a basis for V

(2) $\mathcal{B}$ is linearly indep.

(3) $\operatorname{Span}(\mathcal{B}) = V$.

Proof: Identify V with $\mathbb{R}^n$ (Thm. 4.13 and 4.14) and use the fact that (1), (2), (3) are known to be equivalent in $\mathbb{R}^n$ (e.g. by Thm. 2.7 "the alphabet thm").

Application: From now on, if you would like to check whether an n-element subset of an n-dimensional vector space is a basis, it suffices to check only (1) <u>or</u> (2), you no longer have to check both.

The dimension of $\operatorname{Ker}(A)$ and $\operatorname{Ran}(A)$

Week 7
Page 7

Recall from example $\star$ in week 7, how to find bases of $\operatorname{Ker}(A)$, $\operatorname{Row}(A)$ and $\operatorname{Col}(A)$ of a matrix $A \in \mathbb{R}^{m \times n}$ by using its reduced row echelon form $\tilde{A}$.

4.6. The rank of a matrix

<u>Def.</u> Let $A \in \mathbb{R}^{m \times n}$ and define the rank of A by:

$$\operatorname{rank}(A) := \dim \operatorname{Ran}(A)$$

reduced row echelon form

<u>Thm. 4.19</u> $A \in \mathbb{R}^{m \times n}$ and $\tilde{A}$ its RREF.

Let k be the number of pivots in $\tilde{A}$. Then:

(1) The non-zero rows $\tilde{a}_1, \dots, \tilde{a}_k$ of $\tilde{A}$ form a basis of $\operatorname{Row}(A)$

(2) Let $\tilde{A}_{j_1}, \dots, \tilde{A}_{j_k}$ be the pivot columns of $\tilde{A}$. Then $\{A_{j_1}, \dots, A_{j_k}\}$ is a basis of $\operatorname{Ran}(A)$.

(3) $\dim \operatorname{Ran}(A) = \dim \operatorname{Row}(A) = k$

(4) $\dim \operatorname{Ker}(A) = n - k$

<u>Proof</u> <u>(1)</u> Given a set of vectors $\{v_1 \dots v_k\}$, by the def. of Span, the following do not change their Span:

- replacing v_j by $v_j + \lambda v_i$ for a $\lambda \neq 0$
- replacing v_j by λv_j for a $\lambda \neq 0$
- swapping v_j and v_i

Hence, doing row transformations to A does not change $Row(A)$. So in particular $Row(A) = Row(A$.

The set $\{\tilde{a}_1, \dots, \tilde{a}_k\}$ of non-zero rows of $\tilde{A}$, is obviously a basis of $Row(\tilde{A})$ and hence of $Row(A)$.

<u>(2)</u>: Assume that $\tilde{A}_{j_1}, \dots, \tilde{A}_{j_k}$ are the pivot columns of $\tilde{A}$

$\Rightarrow \{\tilde{A}_{j_1}, \dots, \tilde{A}_{j_k}\}$ is a basis of $Ran(\tilde{A}) = Col(\tilde{A})$.

Let $E_1, \dots, E_r$ be the elem. matrices for row-transformations such that $\underbrace{(E_1 \cdot \dots \cdot E_r)}_{=: E} A = \tilde{A}$

$\Rightarrow$ E is an invertible matrix in $\mathbb{R}^{m \times m}$ and $A = E^{-1} \tilde{A}$

In particular $A_1 = E^{-1} \tilde{A}_1 \quad \dots \quad A_n = E^{-1} \tilde{A}_n \quad (*)$

Note: $(*)$ means that the bij. linear map T defined by E^{-1} maps the colums of $\tilde{A}$ to the colums of A.

Recall: • $Col(\tilde{A}) = Span\{\tilde{A}_1, \dots, \tilde{A}_n\}$

• $\{\tilde{A}_{j_1} \dots \tilde{A}_{j_k}\}$ is a basis of $Col(\tilde{A})$

So by (*) and Thm. 4.14:

$$\underbrace{\{T(\tilde{A}_{j_1}), \dots, T(\tilde{A}_{j_k})\}}_{=\{A_{j_1} \dots A_{j_k}\}} \text{ is a basis of } \underbrace{T(\mathrm{Col}(\tilde{A}))}_{=\mathrm{Col}(A).}$$

(3) This follows from (1) and (2).

(4) Every column in $\tilde{A}$ that has no pivots, gives a free variable and hence a vector for the basis of $\mathrm{Ker}(A)$

$$\left.\begin{array}{l} \cdot\ \#\text{ pivots in } \tilde{A} = k \text{ by (1)} \\ \cdot\ \#\text{ columns in } \tilde{A} = n \end{array}\right\} \Rightarrow \#\text{ columns without pivots} = n-k$$

$\Rightarrow$ $\mathrm{Ker}(A)$ has a basis consisting of $-k$ vectors

$\Rightarrow$ $\dim \mathrm{Ker}(A) = n-k$

$\square$

Cor. 4.20 Let $A \in \mathbb{R}^{m \times n}$, then

- $\mathrm{rank}(A) = \dim \mathrm{Ran}(A) = \dim \mathrm{Col}(A) = \dim \mathrm{Row}(A)$
- $\mathrm{rank}(A) = \#$ indep. rows of $A = \#$ indep. colums of A
- The Rank Theorem:

$$n = \mathrm{rank}(A) + \dim \mathrm{Ker}(A)$$

dim of domain

Ex. • $A = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 2 & 4 & 6 & 8 & 10 & 12 \end{pmatrix}$ Q: $\operatorname{rank}(A) = ?$

Answer: $\operatorname{rank}(A) = 2$

Reason 1: The two rows of A are independent vectors in $\mathbb{R}^6$.

Reason 2: The first two columns of A are independent vectors in $\mathbb{R}^2$. So by Thm. 2.7, they build a basis of $\mathbb{R}^2$ and hence also of the column space.

• If $A \in \mathbb{R}^{3\times 4}$ and $\operatorname{Ker}(A) = \operatorname{Span}\{e_1, e_2\}$

Q: What is the rank of A?

Answer: $\operatorname{rank}(A) = n - \dim \operatorname{Ker}(A) = 4 - 2 = 2$

• Let $A \in \mathbb{R}^{2\times 6}$. Q: What is the smallest possible value for $\dim \operatorname{Ker}(A)$?

Answer: $\dim \operatorname{Ker}(A) = n - \underbrace{\operatorname{rank}(A)}_{\leq 2} \geq 6 - 2 = 4$

$\Rightarrow$ 4 is the smallest possible value for $\dim \operatorname{Ker}(A)$

Cor 4.21 $A \in \mathbb{R}^{n\times n}$ is invertible if and only if

(o) $\det A \neq 0$ (Continuation of Thm 2.7)

(p) $\dim \operatorname{Ker} A = 0$

(q) $\operatorname{Rank} A = n$

4.7. Change of basis

Recall: For a vector space V with a basis $\mathcal{B} = \{b_1, \dots, b_n\}$:

the entries of the coordinate vector $[x]_{\mathcal{B}}$ in the basis $\mathcal{B}$ of an element $x \in V$ are the unique scalars $\lambda_1, \dots, \lambda_n$ satisfying: $\lambda_1 b_1 + \dots + \lambda_n b_n = x$

(Notation: we sometimes also write $\lambda_1 = ([x]_{\mathcal{B}})_1, \dots, \lambda_n = ([x]_{\mathcal{B}})_n$)

Now let $\mathcal{C} = \{c_1, \dots, c_n\}$ be another basis of V.

Q: How to compute $[x]_{\mathcal{C}}$ from $[x]_{\mathcal{B}}$? (In particular if $V \neq \mathbb{R}^n$)

In other words: How do we express a vector that is displayed in terms of a basis $\mathcal{B}$ in terms of another basis $\mathcal{C}$?

Recall: We already know a method in case $V = \mathbb{R}^n$ (see beginning of Week 8)

Now let us work on the case where V and W are not necessarily equal to $\mathbb{R}^n$...

(It will be the same as first identifying V with $\mathbb{R}^n$ then applying the method of the $\mathbb{R}^n$ case.)

Let V be a vector space and let B and C be two bases.

Let $x \in V$ and assume we know $[x]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$.

We want to find $[x]_C = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$.

By def. of $[x]_B$ and $[x]_C$, we have

$$\lambda_1 b_1 + \dots + \lambda_n b_n = x = \mu_1 c_1 + \dots + \mu_n c_n$$

Apply the coordinate map $\varphi(x) = [x]_C$ to both sides of the above equation. In other words: write both sides of the equation in terms of the basis C.) :

$$[\lambda_1 b_1 + \dots + \lambda_n b_n]_C = [\mu_1 c_1 + \dots + \mu_n c_n]_C$$

By linearity of φ (Thm. 4.13)

$$\underbrace{\lambda_1 [b_1]_C + \dots + \lambda_n [b_n]_C}_{= ([b_1]_C, \dots, [b_n]_C) \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}} = \underbrace{\mu_1 \overbrace{[c_1]_C}^{= e_1} + \dots + \mu_n \overbrace{[c_n]_C}^{= e_n}}_{= \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}}$$

__Def.__ Define the change-of-base-matrix from B to C by

$$P_{C \leftarrow B} := ([b_1]_C, \dots, [b_n]_C) \in \mathbb{R}^{n \times n}$$

Then, by the above computation we have: $P_{C \leftarrow B} [x]_B = [x]_C \;\forall x \in V$

Moreover $P_{C \leftarrow B}$ is invertible and hence $P_{C \leftarrow B}^{-1} = P_{B \leftarrow C}$

Proof of invertibility: $\{b_1, \dots, b_n\}$ is a basis of V

$\Rightarrow \{[b_1]_C, \dots, [b_n]_C\}$ is a basis of $\mathbb{R}^n$

$\overset{\text{Cor 4.15}}{\Rightarrow} \det([b_1]_C, \dots, [b_n]_C) \neq 0$

$\Rightarrow P_{C \leftarrow B}$ is invertible.

We have just proven the following Thm:

<u>Thm 4.22</u>

Let V be a vector space and let B and C be two bases: $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$.

Then the matrix $P_{C \leftarrow B} := ([b_1]_C, \dots, [b_n]_C) \in \mathbb{R}^{n \times n}$ is invertible and $P_{C \leftarrow B} [x]_B = [x]_C$ for all $x \in V$.

<u>Rmk</u> If $V = \mathbb{R}^n$, then our old definition of $P_{C \leftarrow B}$ (which is $P_{C \leftarrow B} := P_C^{-1} P_B$) coincides with the new definition (which is $P_{C \leftarrow B} := ([b_1]_C, \dots, [b_n]_C)$).

Ex. $V = \mathbb{P}_2$, $B = \{ \underbrace{1}_{=b_1}, \underbrace{1-x}_{=b_2}, \underbrace{1+x+x^2}_{=b_3} \}$, $C = \{1, 2x+x^2, 1-x-x^2\}$

In order to find $P_{C \leftarrow B}$ we have to find $[b_1]_C, [b_2]_C, [b_3]_C$.

Finding $[b_1]_C$: $b_1 = 1$. We have to find $\lambda_1, \lambda_2, \lambda_3$ s.t. that

$$1 = \lambda_1 \cdot 1 + \lambda_2 (2x+x^2) + \lambda_3 (1-x-x^2)$$

Here the solution is obvious: $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$

$\Rightarrow$ The first column of $P_{C \leftarrow B}$ is $[b_1]_C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Finding $[b_2]_C$: $b_2 = 1-x$. We have to find $\lambda_1, \lambda_2, \lambda_3$ s.t. that

$$1-x = \lambda_1 \cdot 1 + \lambda_2 (2x+x^2) + \lambda_3 (1-x-x^2)$$

Here it's not as trivial to spot the solution.

We can rewrite the above equation as:

$$1-x = (\lambda_1 + \lambda_3) \cdot 1 + (2\lambda_2 - \lambda_3) x + (\lambda_2 - \lambda_3) x^2$$

$$\Rightarrow \begin{cases} 1 = \lambda_1 + \lambda_3 \\ -1 = 2\lambda_2 - \lambda_3 \\ 0 = \lambda_2 - \lambda_3 \end{cases}$$

system of linear equations

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ & 2 & -1 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right)$$ augmented matrix

$\Rightarrow$ ... solve system as usual $\Rightarrow \lambda_1 = 2,\ \lambda_2 = -1,\ \lambda_3 = -1$

(You may want to double check your solutions for $\lambda_1, \lambda_2, \lambda_3$ by plugging it into the original equation

$$1 - x = \lambda_1 \cdot 1 + \lambda_2 (2x + x^2) + \lambda_3 (1 - x - x^2)$$
)

$\Rightarrow$ The second column of $P_{C \leftarrow B}$ is $[b_2] = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

Finding $[b_3]_C$: ... analogous to $[b_2]_C$.

We will get $\lambda_1 = 2,\ \lambda_2 = 0,\ \lambda_3 = -1$

$\Rightarrow$ The third column of $P_{C \leftarrow B}$ is $[b_3]_C = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

In conclusion $$P_{C \leftarrow B} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$