Ex. (1.) $V = \mathbb{R}^2$, $H = \mathbb{R} \times \{0\} = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 0,\ x \in \mathbb{R} \right\}$

$\Rightarrow H = \operatorname{Span}\{e_1\}$ (recall : $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$) $\Rightarrow \{e_1\}$ is a gen. set. of H.

Other generating sets are : $\{-e_1\}, \{2e_1\}, \{-100 e_1\}, \dots$

(2.) $V = \mathbb{R}^3$, H = a plane in $\mathbb{R}^3$ containing the origin

$\Rightarrow \exists\, v, w \in H$ s.t. $H = \operatorname{Span}\{v, w\}$

i.e. $\exists\, u \in H$ can be written as $u = av + bw$

for some scalers $a, b \in \mathbb{R}$

(v, w are the directional vectors of the plane)

Proof of Thm 4.3

(H1) We have to check whether 0_V is in $\operatorname{Span}\{v_1, \dots, v_k\}$.

This means, we have to find coefficients $\lambda_1, \dots, \lambda_k \in \mathbb{R}$

so that $0_V = \lambda_1 \cdot v_1 + \dots + \lambda_k v_k = \sum_{i=1}^{k} \lambda_i v_i$

Choose $\lambda_1 = \dots = \lambda_k = 0 \Rightarrow \sum_{i=1}^{k} \underbrace{0 \cdot v_i}_{= 0_V \text{ Lemma 4.1}} \overset{(A4)}{=} 0_V$

(H2) Assume $u, w \in \operatorname{Span}\{v_1, \dots, v_k\}$

$\Rightarrow u = \sum_{i=1}^{k} \lambda_i v_i$, $w = \sum_{i=1}^{k} \mu_i v_i$ for some real numbers $\lambda_1, \dots \lambda_k, \mu_1, \dots \mu_k \in \mathbb{R}$

$$\Rightarrow \quad u+v = \sum_{i=1}^{k} \lambda_i v_i + \sum_{i=1}^{k} \mu_i v_i \underset{\substack{(A2)(A3)\\(S2)(S3)}}{=} \sum_{i=1}^{k} (\lambda_i v_i + \mu_i v_i)$$

$$\underset{\substack{(A2)(A3)\\(S2)(S3)}}{=} \sum_{i=1}^{k} \underbrace{(\lambda_i + \mu_i)}_{\in \mathbb{R}} v_i$$

$\Rightarrow$ We write $u+v$ as a lin. comb. of v_i

$\Rightarrow$ $u+v \in \mathrm{Span}\{v_1, \dots, v_k\}$

(H3) Similar to (H2).

$\Rightarrow$ So $\mathrm{Span}\{v_1, \dots, v_k\}$ is a subspace of V.

Minimal:

Assume $H \subseteq V$ is a subspace that contains $\{v_1, \dots, v_k\}$.

$\underset{(H2)+(H3)}{\Rightarrow}$ For all $\lambda_1 \dots \lambda_k \in \mathbb{R}$: $\lambda_1 v_1 + \dots + \lambda_k v_k \in H$

$\Rightarrow$ $\mathrm{Span}\{v_1 \dots v_k\} \subseteq H$. $\square$

Generating sets are not unique. Moreover, they can be arbitrarily large but not arbitrarily small ...

Ex. $H = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$ is a subspace and

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a generating set of H.

- Can you name a different generating set for H ?

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \underbrace{\begin{pmatrix}0\\2\\0\end{pmatrix}}_{2\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}} \right\}, \quad \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}} \right\}, \ \dots$$

- Can you name a generating set of H that consists of more than two vectors ?

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \underbrace{\begin{pmatrix}0\\0\\0\end{pmatrix}}_{\text{zero}} \right\}, \quad \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}}_{\text{same}} \right\}, \quad \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \underbrace{\begin{pmatrix}2\\1\\0\end{pmatrix}}_{\text{linear combo}} \right\}$$

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \underbrace{\begin{pmatrix}2\\0\\0\end{pmatrix} \begin{pmatrix}1\\2\\0\end{pmatrix}}_{\text{linear combos}} \right\}$$

↪ Can add as many linear combos as I want

- Can you name a generating set for H that consists of less than two vectors ?

→ No because $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ lin. indep. $\Rightarrow$ H is a plane

We cannot span a plane by one vector.

Def. A set of vectors $\{v_1 \dots v_k\}$ in a vector space V is called *linearly independent* if the only solution $\lambda_1 \dots \lambda_k \in \mathbb{R}$ of the equation $\lambda_1 v_1 + \dots + \lambda_k v_k = 0$ is $\lambda_i = 0 \ \forall i$.

If $\{v_1 \dots v_k\}$ are not lin. indep. we call them *lin. dependent*.

Def. Let V be a vector space. A set of vectors $\mathcal{B} = \{b_1, \dots, b_k\} \subset V$ is called a basis of V if:

(B1) $\text{Span}\{b_1, \dots, b_k\} = V$

(B2) $\{b_1, \dots, b_k\}$ is linearly independent

(Remember: subspaces are vector spaces too!)

Rmk A basis of a vector space is a minimal generating set. Minimal in the sense that if we remove any element of a basis, then it no longer generates the subspace. (This follows from linear independence: non of the bi can be recovered by the others.)

You can also use generating sets (or bases) to prove that a subset $H \subset V$ is a subspace...

Ex. Let $V = \mathbb{R}^3$. Is $H = \left\{ \begin{pmatrix} a-b \\ 2b \\ a+b \end{pmatrix} : a, b \in \mathbb{R} \right\}$ a subspace of V?

$\rightarrow$ Yes! Proof 1: Check the three conditions (H1), (H2), (H3).

Proof 2: $\begin{pmatrix} a-b \\ 2b \\ a+b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow H = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$

and we know from Thm 4.3 that it is a subspace.

4.2. Kernel, Range, column space, row space

Def. Let $A \in \mathbb{R}^{m \times n}$ a matrix. We define the kernel of A by:

$$\operatorname{Ker}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

zero vector in $\mathbb{R}^m$

Thm. 4.4 Let $A \in \mathbb{R}^{m \times n}$. Then $\operatorname{Ker}(A)$ is a subspace of $\mathbb{R}^n$.

Proof. (H1) $A \cdot 0 = 0 \Rightarrow 0 \in \operatorname{Ker}(A)$.

(H2) Assume $x, y \in \operatorname{Ker}(A)$. This means that $Ax = 0$ and $Ay = 0$.

$\Rightarrow A(x+y) = 0 \Rightarrow x+y \in \operatorname{Ker}(A)$.

(H3) Assume $x \in \operatorname{Ker}(A)$, this means that $Ax = 0$

Let $\lambda \in \mathbb{R} \Rightarrow A(\lambda x) = \lambda \cdot Ax = \lambda \cdot 0 = 0$

Rmk. $\operatorname{Ker}(A)$ is the solution space of the homogeneous equation $Ax = 0$.

Def. For a matrix $A \in \mathbb{R}^{m \times n}$. The range (or: the image) of A is:

$$\operatorname{Ran}(A) := \{b \in \mathbb{R}^m : \exists\, x \in \mathbb{R}^n \text{ with } Ax = b\}$$

Def. For a matrix $A \in \mathbb{R}^{m\times n}$ and $A_1, \dots, A_n \in \mathbb{R}^n$ its columns and $a_1^T, \dots, a_m^T$ its rows. We define

$Col(A) = Span\{A_1 \dots A_n\}$ the column space of A

$Row(A) = Span\{a_1, \dots, a_m\}$ the row space of A

Ex. $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$, $Col(A) = Span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$

$Row(A) = Span\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\right\}$

Thm. 4.5 $Ran(A) = Col(A)$ and hence $Ran(A) \subseteq \mathbb{R}^m$ is a subspace.

Proof: $b \in Col(A) \iff b = \underbrace{\lambda_1 A_1 + \dots + \lambda_n A_n}_{= A\cdot\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}}$ for some $\lambda_i \in \mathbb{R}$

$\iff b = Ax$ for some $x \in \mathbb{R}^n$ $\iff b \in Ran(A)$

Subspace follows from Thm 4.3 (Spans are subspaces) $\square$

Ex ✱: Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 1 \end{pmatrix}$. Find $\mathrm{Ker}(A)$, $\mathrm{Row}(A)$ and $\mathrm{Ran}(A)$, and a basis for each of them.

Ker(A): We have to solve $Ax = 0$, so we have to bring A into reduced row echelon form by row transformations.

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \end{pmatrix} \overset{-1}{\sim} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{-1}{\sim} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\text{and swap}}{\overset{-1}{\sim}} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \tilde{A}$$

$\mathrm{Ker}(A) = \{x \in \mathbb{R}^4 : Ax = 0\}$ ⟵ solution space of the equation $Ax = 0$

$$= \left\{ t \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} : t, s \in \mathbb{R} \right\}$$

⟸ $x_4 = t$, $x_3 = s$ free variables and $x_1 = -t - s$, $x_2 = -s$

$$\Rightarrow \mathrm{Ker}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

and $B = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a generating set of $\mathrm{Ker}(A)$.

Since the vectors are independent, it is a basis of $\mathrm{Ker}(A)$.

(This method always yields indep. vectors as every free variable (e.g. x_4) leads to one vector and this vector will carry a 1 in a position where all the vectors of the other free variables have a zero.)

Row(A)

$$\mathrm{Row}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\5\\2\end{pmatrix}\right\} \quad \text{and} \quad \left\{\underbrace{\begin{pmatrix}1\\2\\3\\1\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}1\\1\\2\\1\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}2\\3\\5\\2\end{pmatrix}}_{v_3}\right\}$$

is a generating set. However it is not a basis since:

$v_3 = v_1 + v_2 \Rightarrow v_1, v_2, v_3$ dependent

To turn $\{v_1, v_2, v_3\}$ into a basis, we have (systematically) eliminate the v_i that are linear combinations of other v_i.

But that is the same as row elimination for A, which we have already done!

$\Rightarrow$ The non-zero rows of $\tilde{A} = \begin{pmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{pmatrix}$ are a basis for $\mathrm{Row}(A)$.

$\Rightarrow \mathcal{B} = \left\{\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right\}$ is a basis for $\mathrm{Row}(A)$.

Ran(A) = Col(A)

Bring A into a "(reduced) column echelon form" $\tilde{\tilde{A}}$ by only column transformations:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 3 & 5 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \tilde{\tilde{A}}$$

$\Rightarrow$ $\operatorname{Ran}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis of $\operatorname{Ran}(A)$.

Alternatively

Fact: row transformation preserve dependence, independence of colums. (Proof omitted)

Columns 1 and 2 in the reduced row echelon form $\tilde{A}$ of A are easy to observed to be a basis of $\operatorname{Col}(\tilde{A})$.

$\overset{\text{by Fact}}{\Longrightarrow}$ Columns 1 and 2 of $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \end{pmatrix}$ are a basis of $\operatorname{Col}(A)$.

$\Rightarrow$ $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\}$ is a basis of $\operatorname{Col}(A)$.

⚠ The columns of $\tilde{A}$ are not in $\operatorname{Col}(A)$.

So you cannot pick them as a basis of $\operatorname{Col}(A)$!!!

You have to pick columns of A, but $\tilde{A}$ tells you which ones.

Ex. Is $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ a basis of $\mathbb{R}^3$?

$\to$ Yes!

You could of course prove this by definition, but that is lengthy and tedious.

Short-cut: Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\} = \mathrm{Col}(A) = \mathrm{Ran}(A)$$

By Thm 2.7. and 3.5:

- (B1) $\mathrm{Ran}(A) = \mathbb{R}^3$ iff $\det A \neq 0$.
- (B2) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ are indep. iff $\det A \neq 0$.

So B is a basis of $\mathbb{R}^3$ iff $\det A \neq 0$

We can compute $\det(A) = -1$. So B is a basis of $\mathbb{R}^3$.

Prop. 4.6. $\{v_1, \dots, v_n\} \subseteq \mathbb{R}^n$ is a basis of $\mathbb{R}^n \iff \det(v_1, \dots, v_n) \neq 0$

Proof: Thm. 2.7 and 3.5 as above. $\square$

Kernel and range of a linear transformation

Def. Let V, W be vector spaces. A fct. $f: V \to W$ is a *linear transformation* if:

(i) $f(u+v) = f(u) + f(v) \quad \forall u, v \in V$

(ii) $f(\lambda u) = \lambda f(u) \quad \forall u \in V, \lambda \in \mathbb{R}$

or equivalently, if

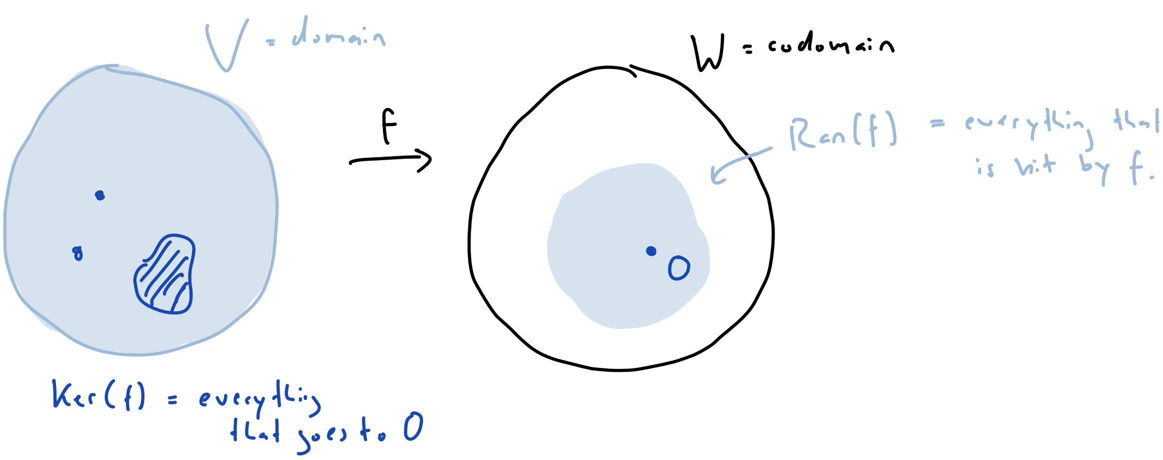
(iii) $f(u + \lambda v) = f(u) + \lambda f(v) \quad \forall u, v \in V, \lambda \in \mathbb{R}$

Moreover, we define the *kernel* and *range* of f:

$$\mathrm{Ker}(f) = \{ v \in V : f(v) = 0_W \}$$

$$\mathrm{Ran}(f) = \{ w \in W : f(v) = w \text{ for some } v \in V \}$$

Schematic illustration:



Thm 4.7 Let V, W be vector spaces and $f: V \to W$ a linear transformation then:

- $\mathrm{Ker}(f)$ is a subspace of V
- $\mathrm{Ran}(f)$ is a subspace of W

(Proof: omitted since exactly as seen before twice.)

Lemma 4.8 Let V, W be vector spaces and $f: V \to W$ a linear transformation then:

(i) f is surjective if and only if $\mathrm{Ran}(f) = W$

(ii) f is injective if and only if $\mathrm{Ker}(f) = \{0\}$.

Proof (i) follows from the definitions of surjectivity and $\mathrm{Ran}(f)$.

(ii) $\boxed{\Rightarrow}$ f injective

$\Rightarrow$ For each element in $w \in W$, there is at most one element $v \in V$ so that $f(v) = w$.

Since f is linear: $f(0_V) = f(0 \cdot 0_V) = 0 \cdot f(0_V) = 0_W$

$\Rightarrow$ So $v = 0_V$ is the only $v \in V$ so that $f(v) = 0_W$

$\Rightarrow$ $\mathrm{Ker}(f) = \{0_V\}$

$\boxed{\Leftarrow}$ Assume $\mathrm{Ker}(f) = \{0_V\}$

Let $v, \tilde{v} \in V$ so that $f(v) = f(\tilde{v})$. (To show: $v = \tilde{v}$.)

$$0_W = f(v) - f(\tilde{v}) \underset{\uparrow f \text{ linear}}{=} f(v - \tilde{v}) \Rightarrow v - \tilde{v} \in \mathrm{Ker}(f)$$

By assumption $\mathrm{Ker}(f) = \{0\}$, so $v - \tilde{v} = 0_V \Rightarrow v = \tilde{v}$ $\square$

Some examples of bases:

- $\left\{ \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{=e_1}, \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{=e_2}, \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{=e_3} \right\}$ is a basis for $\mathbb{R}^3$.

  We call it the standard basis of $\mathbb{R}^3$.

- $\left\{ \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix} \right\}$ is the standard basis of $\mathbb{R}^n$

- Can you give a different basis for $\mathbb{R}^3$?

  $\left\{ \begin{pmatrix}2\\0\\0\end{pmatrix} \begin{pmatrix}0\\2\\0\end{pmatrix} \begin{pmatrix}0\\0\\2\end{pmatrix} \right\}, \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}, \dots$

- Does the matrix space $\mathbb{R}^{2\times 2}$ have a basis?

  (What would be a good standard basis?)

  $\mathbb{R}^{2\times 2} = \left\{ \begin{pmatrix}a & b\\c & d\end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}$

  $\mathcal{B} = \left\{ \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \right\}$ is a basis, we call it the standard basis of $\mathbb{R}^{2\times 2}$

  $\tilde{\mathcal{B}} = \left\{ \begin{pmatrix}-\pi & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-3 & 1\\2 & 4\end{pmatrix}, \begin{pmatrix}-3 & 0\\2 & 4\end{pmatrix} \right\}$ is a basis too.

- What about $\mathbb{R}^{m\times n}$?

$$\mathcal{B} = \left\{ \begin{matrix} \begin{pmatrix}1 & 0 & \cdots & 0\\0 & \cdots & \cdots & 0\\\vdots & & & \vdots\\0 & \cdots & \cdots & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0 & \cdots & 0\\0 & \cdots & \cdots & & 0\\\vdots & & & & \vdots\\0 & \cdots & \cdots & & 0\end{pmatrix} \dots \begin{pmatrix}0 & \cdots & 0 & 1\\0 & \cdots & \cdots & 0\\\vdots & & & \vdots\\0 & \cdots & \cdots & 0\end{pmatrix}, \\ \begin{pmatrix}0 & \cdots & \cdots & 0\\1 & 0 & \cdots & 0\\\vdots & & & \vdots\\0 & \cdots & \cdots & 0\end{pmatrix}, \begin{pmatrix}0 & \cdots & \cdots & & 0\\0 & 1 & 0 & \cdots & 0\\\vdots & & & & \vdots\\0 & \cdots & \cdots & & 0\end{pmatrix} \dots \begin{pmatrix}0 & \cdots & \cdots & 0\\0 & \cdots & 0 & 1\\\vdots & & & \vdots\\0 & \cdots & \cdots & 0\end{pmatrix}, \\ \vdots \\ \begin{pmatrix}0 & \cdots & \cdots & 0\\\vdots & & & \vdots\\0 & \cdots & \cdots & 0\\1 & 0 & \cdots & 0\end{pmatrix}, \begin{pmatrix}0 & \cdots & \cdots & & 0\\\vdots & & & & \vdots\\0 & \cdots & \cdots & & 0\\0 & 1 & 0 & \cdots & 0\end{pmatrix} \dots \begin{pmatrix}0 & \cdots & \cdots & 0\\\vdots & & & \vdots\\0 & \cdots & \cdots & 0\\0 & \cdots & 0 & 1\end{pmatrix} \end{matrix} \right\}$$

is a basis, and we call it the standard basis of $\mathbb{R}^{m\times n}$

Spaces of polynomials

Def. A polynomial is an linear combination of powers of a placeholder variable x:

$$p = a_0 + a_1 x + \ldots + a_n x^n$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ (coefficients), $n \in \mathbb{N}$.

So, basically (for now), a polynomial is a list of coefficients a_i. In particular, two polynomials are identical, if their respective coefficients are identical.

Def. If $a_n \neq 0$: n is the degree of $p(x)$: $\deg(p) = n$

If $p = 0$ $(a_i = 0 \ \forall i)$: $\deg(p) := -\infty$.

Ex.
- $p = 1 + \pi x - 5x^2 \Rightarrow \deg(p) = 2$
- $p = x^{17} + 2x + \sin(2) \Rightarrow \deg(p) = 17$
- $p = 5 \Rightarrow \deg(p) = 0$
- $p = 0 \Rightarrow \deg(p) = -\infty$
- $p = x^3 \Rightarrow \deg(p) = 3$
- $p = \sin(x)\, x^2 + 3x \Rightarrow p$ is not a polynomial

Def. • Let $\mathbb{P}$ be the space of all polynomials

$$\mathbb{P} = \{ a_0 + a_1 x + \dots + a_n x^n : n \in \mathbb{N}_0,\ a_i \in \mathbb{R} \}$$

• For $d \in \mathbb{N}$, let $\mathbb{P}_d$ be the space of polynomials of degree $\leq d$:

$$\mathbb{P}_d = \{ a_0 + a_1 x + \dots + a_n x^n : n \in \mathbb{N}_0,\ n \leq d,\ a_i \in \mathbb{R} \}$$

Ex • $x^3 + x \in \mathbb{P}_3$ ✓

• $2x \in \mathbb{P}_4$ ✓

• $5x^4 + 1 \cancel{\in} \mathbb{P}_3$

• $1 \in \mathbb{P}_2$ ✓

• $x \in \mathbb{P}_4$ ✓

• all of the previous examples are in $\mathbb{P}_4$.

• all of the previous examples are in $\mathbb{P}$.

• $0 \in \mathbb{P}_n$, $0 \in \mathbb{P}$ ✓.

Prop. 4.9 $\mathbb{P}$ is a vector space and $\mathbb{P}_d \subseteq \mathbb{P}$ a subspace.

(Proof: easy by definition)

Homework 7: Many examples of subspaces of $\mathbb{P}$ or $\mathbb{P}_n$.

Def. For a polynomial $p = a_0 + a_1 x + .. + a_n x^n \in \mathbb{P}$, we define its derivative p' by:

$$p' = \underbrace{a_1} + \underbrace{2a_2} x + ... + \underbrace{n a_n} x^{n-1} \in \mathbb{P}$$

Consider the transformation $T : \mathbb{P} \to \mathbb{P}$ given by: $T(p) := p'$

Then, one can easily check: For $p, q \in \mathbb{P}$, $\lambda \in \mathbb{R}$

- $(p+q)' = p' + q'$
- $(\lambda p)' = \lambda p'$

Hence T is a linear transformation.

Q: $\operatorname{Ker}(T) = ?$ $\quad \operatorname{Ran}(T) = ?$

Ker(T): Let $p \in \operatorname{Ker}(T) \Rightarrow p \in \mathbb{P}$ and $T(p) = 0$

$p = a_0 + a_1 x + .. + a_n x^n$

the zero polynomial: $0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 ...$

$T(p) = p' = a_1 + 2a_2 x + .. + n a_n x^{n-1} = 0$

$\Rightarrow a_1 = 2a_2 = ... = n a_n = 0 \Rightarrow a_1 = a_2 = .. = a_n = 0$

two polyn. are the same if their coeff. are the same "comparison of coeff."

$\Rightarrow p = a_0 + 0 = a_0 \Rightarrow p \in \mathbb{P}_0$

space of polyn. of degree ≤ 0 (i.e. space of constants)

So we have $\operatorname{Ker}(T) \subseteq \mathbb{P}_0$.

Do we also have $\mathbb{P}_0 \subseteq \operatorname{Ker}(T)$?

Let $p \in \mathbb{P}_0 \Rightarrow p = a_0 \Rightarrow T(p) = T(a_0) = 0$

$\Rightarrow \mathbb{P}_0 \subseteq Ker(T)$ and hence $\mathbb{P}_0 = Ker(T)$.

Ran(T) : which polynomials can appear as a derivate of another polynomial ? $\rightarrow$ Every polynomial can! So...

Claim: $Ran(T) = \mathbb{P}$.

Proof: Clearly $Ran(T) \subseteq \mathbb{P}$.

So it suffices to check : $\mathbb{P} \subseteq Ran(T)$.

Let $p = a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{P}$

Define $q := c + a_0 x + \frac{1}{2} a_1 x^2 + \ldots + \frac{1}{n+1} a_n x^{n+1}$

$\Rightarrow T(q) = p \Rightarrow p \in Ran(T)$ $\square$

Rmk. Every polynomial can be viewed as a polynomial function.

by definition, just list of coefficients a_i written in a fancy way

To formally justify this identification, we must check:

For $p, q \in \mathbb{P}$: $\underbrace{\underset{\text{as polynomials}}{p = q}}_{\text{coefficient are the same}} \iff \underbrace{\underset{\text{as functions}}{p = q}}_{p(x) = q(x)\ \forall x \in \mathbb{R}}$

<u>Proof</u> Let $p, q \in \mathbb{P}$

$\Rightarrow$ We can write: $p = a_0 + a_1 x + .. + a_n x^n$

$q = b_0 + b_1 x + .. + b_m x^m$

$\boxed{\Rightarrow}$ Assume $p = q$ as polynomials

$\Leftrightarrow a_0 = b_0, .., a_n = b_n, m = n$

$\Rightarrow p(x) = q(x) \;\forall x \in \mathbb{R} \Rightarrow p = q$ as functions

$\boxed{\Leftarrow}$ Assume $p = q$ as functions

$\Rightarrow p(x) = q(x) \;\forall x \in \mathbb{R}$. (To show: $a_i = b_i \;\forall i$)

$\Rightarrow a_0 + a_1 x + .. + a_n x^n = b_0 + b_1 x + .. + b_m x^m$

for all $x \in \mathbb{R}$

Plugging $x = 0$ into the above equation yields: $a_0 = b_0$

Next, if $p(x) = q(x) \;\forall x$ then also $p'(x) = q'(x) \;\forall x \in \mathbb{R}$

$\Rightarrow a_1 + 2a_2 x + ... + n a_n x^{n-1} = b_1 + 2b_2 x + .. + m b_m x^{m-1}$

for all $x \in \mathbb{R}$

Again, plug in $x = 0 \Rightarrow a_1 = b_1$

Continue derivating and plugging in $x = 0$ yields: $a_i = b_i \;\forall i$

$\square$

<u>Rmk.</u> Notice that also our definition of the derivative of a polyn. coincides with what the derivative (as defined in analysis) of a polyn. fct. is.

Linear independence and bases for polynomial spaces

Recall : $\mathbb{P}_n = \{ a_0 + a_1 x + \dots + a_n x^n : a_i \in \mathbb{R}, n \in \mathbb{N}_0 \}$

What could be a basis of $\mathbb{P}_n$?

Candidate : $\{1, x, x^2, \dots, x^n\}$. → Let's check!

Generating set By definition of $\mathbb{P}_n$, every element of $\mathbb{P}_n$

is a linear combination of $1, x, \dots, x^n$.

So $\{1, x, x^2, \dots, x^n\}$ is a generating set of $\mathbb{P}_n$.

Independence : Let $\lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}$ and consider the

equation $\lambda_0 \cdot 1 + \lambda_1 x + \dots + \lambda_n x^n = 0$ the zero polynomial

(So, the above equation can also be written:

$$\lambda_0 \cdot 1 + \lambda_2 x + \dots + \lambda_n x^n = 0 \cdot 1 + 0 \cdot x + \dots + 0 \cdot x^n$$)

Comparison of coeff. $\Rightarrow \lambda_1 = 0 \ \dots \ \lambda_n = 0$

$\Rightarrow 1, x, \dots x^n \in \mathbb{P}_n$ independent.

$\Rightarrow \{1, x, x^2, \dots, x^n\}$ is a basis of $\mathbb{P}_n$

Q : Can we find a basis for $\mathbb{P}$?

→ Yes! The infinite (but countable set) $\{1, x, x^2, \dots\}$ is a basis for $\mathbb{P}$. To formally grasp this we need to have the correct concept of lin. independence and and of spans for infinite sets of vectors (such as $\{1, x, x^2, \dots\}$).

Thm 4.10 $\{1, x, \dots, x^n\}$ is a basis of $\mathbb{P}_n$

$\{1, x, x^2, \dots\}$ is a basis of $\mathbb{P}$.

We call them the standard basis of $\mathbb{P}_n$ resp. $\mathbb{P}$.

Infinite generating sets

not relevant for exam

Def. Let V be a vector space and $S \subseteq V$ an infinite subset.

The set S of vectors is lin. indep. if for every finite subset $\{v_1, \dots, v_k\}$ of S:

$$\lambda_1 v_1 + \dots + \lambda_k v_k = 0 \implies \lambda_1 = \dots = \lambda_k = 0$$

Def. The Span of an infinite subset $S \subseteq V$ are all finite linear combinations of elements in S:

$$\mathrm{Span}(S) = \{\lambda_1 s_1 + \dots + \lambda_k s_k : k \in \mathbb{N},\ s_i \in S,\ \lambda_i \in \mathbb{R}\}$$

Consider the space of polynomials $\mathbb{P}$.

Claim: $B := \{1, x, x^2, x^3, \dots\} = \{x^n : n \in \mathbb{N}_0\}$ is a basis for $\mathbb{P}$.

Proof: $\operatorname{Span}(B) = \mathbb{P}$: All elements of B are polynomials.

$\Rightarrow B \subseteq \mathbb{P} \Rightarrow \operatorname{Span}(B) \subseteq \mathbb{P}$.

since $\mathbb{P}$ is a vector space

Let $p \in \mathbb{P} \Rightarrow$ $p = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
for some coeff. $a_1, \dots, a_n \in \mathbb{R}$, $n \in \mathbb{N}$

$\Rightarrow p$ is a finite linear comb. of elements in B

$\Rightarrow p \in \operatorname{Span}(B)$

Hence: $\mathbb{P} \subseteq \operatorname{Span}(B)$

linear independence: Let $\{p_1, \dots, p_k\} \subseteq B$ a finite subset.

Assume that $\lambda_1 p_1 + \dots + \lambda_k p_k = 0$ ← The zero polynomial (i.e. the constant zero fct.)

By comp. of coeff. $\Rightarrow \lambda_1 = \dots = \lambda_k = 0$

$\Rightarrow \{p_1, \dots, p_k\}$ lin. indep.

The above argument holds for all finite subsets $\{p_1, \dots, p_k\} \subseteq B$

$\Rightarrow B$ lin. indep.

$\Rightarrow \mu_0 = \dots = \mu_n = 0$ ← since two polyn. are the same if and only if all coeff. are the same.

$\Rightarrow \lambda_1 = \dots = \lambda_k = 0$.

$\square$

Bases of Spans (finite again)

Thm 4.11 Let V be a vector space that can be written as $V = \text{Span}\{v_1, \dots, v_k\}$. Then:

(1) If a v_j is a linear comb. of other v_is then
$$V = \text{Span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}$$

(2) If $V \neq \{0_V\}$, then some subset of $\{v_1, \dots, v_k\}$ forms a basis of V.

This theorem applies e.g. to vector spaces: $\mathbb{R}^n$, $\mathbb{R}^{n \times m}$, $\mathbb{P}_n$ as well as most spaces that you have seen in the homework.
Exceptions: $\mathbb{P}$, $\{f : \mathbb{R}^n \to \mathbb{R}^n : f \text{ continuous}\}$, and similar.

Proof: (1) follows from the definition of a Span itself.
(2) is a simple consequence of (1). $\square$

Thm 4.11 is the reason why the method for finding bases of $\text{Row}(A)$ and $\text{Col}(A)$ that are explained in Example ✱ always work.

4.4. Coordinate systems

<u>Thm. 4.12</u> Let V be a vectorspace and $B = \{b_1, \dots, b_n\}$ a basis for V. Then for each $v \in V$ there <u>exist</u> <u>unique</u> scalars $\lambda_1, \dots \lambda_n \in \mathbb{R}$ such that:

$$v = \lambda_1 b_1 + .. + \lambda_n b_n$$

<u>Proof</u> The scalars $\lambda_1, \dots \lambda_n \in \mathbb{R}$ <u>exist</u> since $\operatorname{Span} B = V$.

They are <u>unique</u> because $\{b_1, \dots, b_n\}$ are lin. indep.:

Assume, there are also $\mu_1, \dots, \mu_n \in \mathbb{R}$ such that:

$v = \mu_1 b_1 + .. + \mu_n b_n$.

$$\begin{aligned} \Rightarrow \quad 0 &= v - v \\ &= (\lambda_1 b_1 + .. + \lambda_n b_n) - (\mu_1 b_1 + .. + \mu_n b_n) \\ &= (\lambda_1 - \mu_1) b_1 + .. + (\lambda_n - \mu_n) b_n. \end{aligned}$$

linear indep.

$$\Rightarrow \quad \lambda_1 - \mu_1 = 0, \dots, \lambda_n - \mu_n = 0$$

$$\Rightarrow \quad \lambda_1 = \mu_1, \dots, \lambda_n = \mu_n \qquad \square$$

Def. Let V be a vectorspace and $B = \{b_1, \dots, b_n\}$ an ordered basis. Let $v \in V$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be the unique scalars with $v = \lambda_1 b_1 + \dots + \lambda_n b_n$.

We call the vector
$$[v]_B := \begin{pmatrix} \lambda \\ \vdots \\ \lambda_n \end{pmatrix}$$
the coordinate vector of v in the basis B.

Obviously, the order of the vectors $b_1, \dots, b_n$ in B matters. That is why we require B to be an ordered basis.

e.g. Consider e_1 and e_2 in $\mathbb{R}^2$. $B = \{e_1, e_2\}$ and $\tilde{B} = \{e_2, e_1\}$ are the same bases but not the same ordered bases.

Rmk For all the standard bases introduced so far, we consider the usual ordering the standard case:

e.g. $\{e_1, \dots, e_n\}$ for $\mathbb{R}^n$,

$\{1, x, x^2 \dots\}$ for $\mathbb{P}$ and $\{1, x, \dots, x^n\}$ for $\mathbb{P}_n$

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ for $\mathbb{R}^{2\times 2}$

Ex. A $\quad V = \mathbb{R}^2, \quad B = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}.$

Since $\det\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = 1 \neq 0 \implies B$ is a basis of $\mathbb{R}^2$.

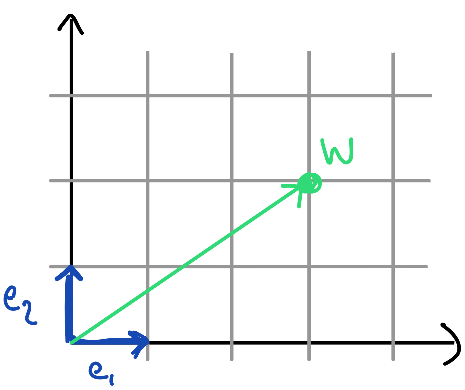
Q: What are the coordinate vectors for $v = \begin{pmatrix}1\\1\end{pmatrix}$ and for $w = \begin{pmatrix}3\\2\end{pmatrix}$ in the basis B?

Answer:

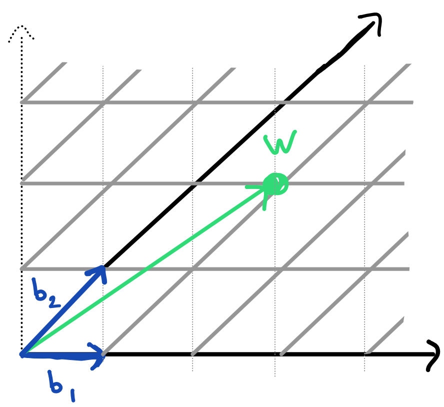
$$v = \begin{pmatrix}1\\1\end{pmatrix} = 0 \cdot \begin{pmatrix}1\\0\end{pmatrix} + 1 \cdot \begin{pmatrix}1\\1\end{pmatrix} \implies [v]_B = \begin{pmatrix}0\\1\end{pmatrix}$$

$$w = \begin{pmatrix}3\\2\end{pmatrix} = 1 \cdot \begin{pmatrix}1\\0\end{pmatrix} + 2 \cdot \begin{pmatrix}1\\1\end{pmatrix} \implies [v]_B = \begin{pmatrix}1\\2\end{pmatrix}$$

Geometric interpretation: $\quad V = \mathbb{R}^2, \quad B = \left\{ \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{=b_1}, \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{=b_2} \right\}, \quad w = \begin{pmatrix}3\\2\end{pmatrix}$



Grid given by standard basis



one step with b_1

$$[w]_B = \begin{pmatrix}1\\2\end{pmatrix}$$

two steps with b_2

Grid given by $B = \left\{ \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{=b_1}, \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{=b_2} \right\}$

Ex. B $V = \mathbb{R}^3$, $\mathcal{B} = \{e_1, e_2, e_3\}$, $v = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}$, $[v]_{\mathcal{B}} = ?$

$$v = -1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow [v]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} = v$$

Rmk This of course holds in general:

For a vector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ the coordinate vector of v in the standard basis is $[v]_{\mathcal{B}} = v$ since

$$v = v_1 e_1 + \ldots + v_n e_n.$$

Ex. C $V = \mathbb{P}_2$, $\mathcal{B} = \{1, x, x^2\}$

$p = 2x^2 + x - 5 \in \mathbb{P}_2 \Rightarrow [p]_{\mathcal{B}} = ?$

$$p = -5 \cdot 1 + 1 \cdot x + 2x^2 \Rightarrow [p]_{\mathcal{B}} = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}$$

A different basis: $\tilde{\mathcal{B}} = \{1, x^2, 1+x\}$

$\Rightarrow [p]_{\tilde{\mathcal{B}}} = ?$

$$p = -6 \cdot 1 + 2x^2 + 1 \cdot (1+x) \Rightarrow [p]_{\mathcal{B}} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$$

$\underset{\sim}{Q}$: Is there a method that helps us to quickly find the representation of a vector in a desired basis?

$\underset{\sim}{Q}$: If we already have the representation of a vector in some basis, can we just "translate" to a different basis?

→ We will start answering these questions for the case where $V = \mathbb{R}^n$.

Let us go back to Example A:

$w = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ expressed in the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is $[w]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Because: $2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

This can be rewritten as: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Let M be the matrix whose columns are the vectors in $\mathcal{B}$ (in order!) then: $M [w]_{\mathcal{B}} = w$

Since M is invertible, we can also write this as:

$$M^{-1} w = [w]_{\mathcal{B}}$$

Coordinates and coordinate change in $\mathbb{R}^n$

Def. Let $B = \{b_1, \dots, b_n\}$ be an ordered basis of $\mathbb{R}^n$.

We call the matrix $P_B := (b_1, \dots, b_n) \in \mathbb{R}^{n \times n}$

the change of coordinate matrix from B to the standard basis.

Reason: If $x \in \mathbb{R}^n$ then $x = \underbrace{P_B \cdot [x]_B}_{= x_1 b_1 + \dots + x_n b_n}$

So multiplying P_B to a vector written in basis B, yields the vector written in the standard basis.

Since B is a basis, its vectors are independent. Thus, $\det P_B \neq 0$ and P_B is invertible.

$$\Rightarrow \quad P_B^{-1} x = [x]_B \quad \forall x \in \mathbb{R}^n$$

Def. We call P_B^{-1} the change of coordinate matrix from the standard basis to B.

We can also switch between two bases $\mathcal{B}$ and $\tilde{\mathcal{B}}$ of $\mathbb{R}^n$, neither of which is the standard basis:

For $x \in \mathbb{R}^n$:

$$\underbrace{\left(P_{\tilde{\mathcal{B}}}^{-1} \cdot P_{\mathcal{B}}\right)}_{=:\, P_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}}} [x]_{\mathcal{B}} = P_{\tilde{\mathcal{B}}}^{-1} \big(\underbrace{P_{\mathcal{B}} [x]_{\mathcal{B}}}_{= x}\big) = P_{\tilde{\mathcal{B}}}^{-1} x = [x]_{\tilde{\mathcal{B}}}$$

<u>Def.</u> $P_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}}$ is the change of coordinate matrix from $\mathcal{B}$ to $\tilde{\mathcal{B}}$.

<u>Ex.</u> $\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \right\}$, $\tilde{\mathcal{B}} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$

are bases of $\mathbb{R}^3$. Let $w \in \mathbb{R}^3$ so that $[w]_B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Find $[w]_{\tilde{\mathcal{B}}}$.

<u>Solution</u>: $P_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 4 & 0 \\ 3 & 9 & -3 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}}_{= \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 4 & 1/3 \\ 0 & 4 & 2/3 \\ -3 & 9 & 1 \end{pmatrix}$

$$\Rightarrow [w]_{\tilde{\mathcal{B}}} = P_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}} [w]_B = \begin{pmatrix} 1 & 4 & 1/3 \\ 0 & 4 & 2/3 \\ -3 & 9 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 2/3 \\ -2 \end{pmatrix}$$

The coordinate mapping

Thm. 4.13 Let V be a vector space and $\mathcal{B} = \{b_1, \dots, b_n\}$ a basis for V. Define the mapping $T : V \to \mathbb{R}^n$ by:

$$T(v) := [v]_{\mathcal{B}}$$

This map T is linear, bijective and its inverse is linear as well.

Notice: In the case when $V = \mathbb{R}^n$, we already know this to be true as then $T(v) = P_{\mathcal{B}}^{-1} \cdot v$.

In the case where $V \neq \mathbb{R}^n$ (but e.g. V is a polynomial space) Thm. 4.12 allows us to from now on identify that space with $\mathbb{R}^n$.

e.g. $\mathbb{P}_2$ with basis $\mathcal{B} = \{1, x, x^2\}$. $\Rightarrow n = 2$

and we identify $\mathbb{P}_2$ with $\mathbb{R}^3$ by:

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

So we can think of polyn. as actual vectors!

This will have many useful applications!

So let us prove the Thm.

Proof 1.) Linearity We have to show :

$$[x+\lambda y]_B = [x]_B + \lambda [y]_B$$

Let $x = \alpha_1 b_1 + .. + \alpha_n b_n$ and $y = \beta_1 b_1 + .. + \beta_n b_n$

$$\Rightarrow \quad x + \lambda y = (\alpha_1 b_1 + .. + \alpha_n b_n) + \lambda(\beta_1 b_1 + .. + \beta_n b_n)$$

$$= (\alpha_1 + \lambda \beta_1) b_1 + .. + (\alpha_n + \lambda \beta_n) b_n$$

calculus rules in V

$$\Rightarrow [x]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad [y]_B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \quad [x+\lambda y]_B = \begin{pmatrix} \alpha_1 + \lambda \beta_1 \\ \vdots \\ \alpha_n + \lambda \beta_n \end{pmatrix}$$

Hence by $\mathbb{R}^n$ calculus : $[x+\lambda y]_B = [x]_B + \lambda [y]_B$

2.) Injectivity We have to show : If $[v]_B = 0$ then $v = 0$

If $[v]_B = 0 \quad \Rightarrow \quad v = \underbrace{0 \cdot b_1}_{= 0_V} + .. + \underbrace{0 \cdot b_n}_{= 0_V} = 0_V.$

calculus in V

3.) Surjectivity We have to show : For every $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

we have to find $v \in V$ so that $[v]_B = x$.

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$. Define $v = x_1 b_1 + .. + x_n b_n$

$\Rightarrow \quad [v]_B = x.$

4.) Linearity of f^{-1}

<u>Claim</u>: If a linear map $T: V \to W$ is invertible, then $T^{-1}: W \to V$ is also linear.

<u>Proof</u>: Let $w_1, w_2 \in W$, $\lambda \in \mathbb{R}$.

To show: $T^{-1}(w_1 + \lambda w_2) = T^{-1}(v_1) + \lambda T^{-1}(w_2)$.

Define $v_1 := T^{-1}(w_1)$, $v_2 := T^{-1}(w_2)$

$\Rightarrow$ $w_1 = T(v_1)$, $w_2 = T(v_2)$

Hence: $T^{-1}(w_1 + \lambda w_2) = T^{-1}(T(v_1) + \lambda T(v_2))$

$$\underset{\uparrow \atop T \text{ linear}}{=} T^{-1}(T(v_1 + \lambda v_2)) \underset{\uparrow \atop T^{-1}T = id}{=} v_1 + \lambda v_2$$

$$= T^{-1}(w_1) + \lambda T^{-1}(w_2).$$

$\square$

<u>Cor. 4.14</u> Let V be a vector space and $B = \{b_1, \dots, b_n\}$ a basis. Let $S = \{v_1, \dots, v_k\} \subseteq V$ be a set of vectors and let $\tilde{S} = \{[v_1]_B, \dots, [v_k]_B\} \subseteq \mathbb{R}^n$ be the coordinate vectors of the elements of S for the basis B. Then:

(a) S is lin. indep. (in V) if and only if $\tilde{S}$ is lin. indep. (in $\mathbb{R}^n$)

(b.) $\text{Span}(S) = V$ if and only if $\text{Span}(\tilde{S}) = \mathbb{R}^n$.

In conclusion: S is a basis for V if and only if $\tilde{S}$ is a basis for $\mathbb{R}^n$.

<u>Proof.</u> .. ..

Find the Thm that says "A Bij. v_i lin indep

$\Rightarrow Av_i$ indep"

Here is an application of Cor. 4.14:

Ex. $V = \mathbb{P}_2$, $S = \{1+2x^2, 4+x+5x^2, 3+2x\}$

Q: Is S a basis for $\mathbb{P}_2$? (Use Cor. 4.14!)

Answer: $B = \{1, x, x^2\}$ is a basis of $\mathbb{P}_2$

$$[1+2x^2]_B = \begin{pmatrix}1\\0\\2\end{pmatrix}, \ [4+x+5x^2]_B = \begin{pmatrix}4\\1\\5\end{pmatrix}, \ [3+2x]_B = \begin{pmatrix}3\\2\\0\end{pmatrix}$$

So we know by Cor. 4.14:

S is a basis for $\mathbb{P}_2$ iff and only if

$\tilde{S} = \left\{\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}4\\1\\5\end{pmatrix}, \begin{pmatrix}3\\2\\0\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

Is $\tilde{S}$ a basis for $\mathbb{R}^3$?

$$\begin{pmatrix}1 & 4 & 3\\ 0 & 1 & 2\\ 2 & 5 & 0\end{pmatrix} \sim \dots \underset{\text{reduction}}{\text{row}} \sim \begin{pmatrix}1 & 4 & 3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix} \rightarrow \text{Not a basis!}$$

So S is not a basis for $\mathbb{P}_2$.

Add application about maps! → Graded Exercise!