Warm up:

Ex How to compute a determinant by definition.

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & -3 & -1 \end{pmatrix} \qquad \det A = ?$$

$$\det A = 2 \cdot \underbrace{\det\begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}}_{= 1\cdot(-1) - 0\cdot(-3) = -1} + 0 \cdot \ldots + 1 \cdot \underbrace{\det\begin{pmatrix} 0 & 1 \\ 5 & -3 \end{pmatrix}}_{0\cdot(-3) - 1\cdot 5 = -5} = -2 - 5 = -7$$

Ex. Compute determinants using short cuts.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix} \quad E = \begin{pmatrix} -2 & 0 & 1 \\ 3 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -1 & -3 \end{pmatrix} \rightsquigarrow A$$

$\det B = 1$

Because B is the identity matrix

$\det C = 0$

Because two identical columns

$\det E = 0$

Because the second column is a linear combin. of the other columns.

$\det F = -\det A$
$= 7$

Ex

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & d_{nn} \end{pmatrix} \quad \det D = d_{11} \cdot \underbrace{\det\begin{pmatrix} d_{22} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{pmatrix}}_{= d_{22} \cdot \det\begin{pmatrix} d_{33} & & \\ & \ddots & \\ & & d_{nn} \end{pmatrix} + 0 - 0 \ldots} + 0 - 0 \ldots$$

continue...

$$\Rightarrow \det D = d_{11} d_{22} \ldots d_{nn}$$

The determinants of elementary matrices

Next goal: $\det AB = \det A \cdot \det B$

$\rightarrow$ need elementary matrices and their determinants.

Recall: A matrix $E \in \mathbb{R}^{n \times n}$ is called elementary if it is among the following list:

$$D_i(\lambda) = \begin{array}{cc} \begin{matrix} & & i & & \end{matrix} & \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} & \begin{matrix} \\ \\ \leftarrow i \\ \\ \\ \end{matrix} \end{array} \qquad L_{ij}(\lambda) = \begin{array}{cc} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & \lambda & 1 & \\ & & & 1 \end{pmatrix} & \begin{matrix} \\ \\ \leftarrow i \\ \\ \end{matrix} \\ \begin{matrix} & \uparrow j & & \end{matrix} & \end{array}$$

$$P_{ij} = \begin{array}{cc} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} & \begin{matrix} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{matrix} \\ \begin{matrix} & & \uparrow i & & \uparrow j & & \end{matrix} & \end{array} \qquad E \cdot M$$

If we multiply an elementary matrix $E \in \mathbb{R}^{n \times n}$ from the left to a matrix $M \in \mathbb{R}^{n \times d}$ (i.e. $E \cdot M$) then:

- $E = P_{ij} \in \mathbb{R}^{n \times n}$ interchanges rows i and j of M
- $E = D_i(\lambda) \in \mathbb{R}^{n \times n}$ multiplies row i by λ $(\lambda \in \mathbb{R}, \lambda \neq 0)$ of M
- $E = L_{ij}(\lambda) \in \mathbb{R}^{n \times n}$ adds λ times j-th row to the i-th row of M

(Elementary row operations!)

Lemma 3.3 $\det D_i(\lambda) = \lambda$, $\det P_{ij} = -1$, $\det L_{ij}(\lambda) = 1$

Proof

- $D_i(\lambda)$ is a diagonal matrix. So by the above example:

  $\det D_i(\lambda) = 1 \cdot \ldots 1 \cdot \lambda \cdot 1 \ldots \cdot 1 = \lambda$

- P_{ij} is obtained from I_n by swapping the i-th and j-th row

  $\Rightarrow \det P_{ij} = -\det I_n = -1$

Recall (from Thm 3.1): $f(x) = \det \underbrace{(A_1 \ldots x \ldots A_n)}$ is linear

the matrix A where the k-th column is replaced by the vector x.

This means:

- $f(x+y) = f(x) + f(y)$

  $\Rightarrow \det(A_1 \ldots x+y \ldots A_n) = \det(A_1 \ldots x \ldots A_n) + \det(A_1 \ldots y \ldots A_n)$

- $f(\lambda x) = \lambda f(x)$

  $\Rightarrow \det(A_1 \ldots \lambda x \ldots A_n) = \lambda \det(A_1 \ldots x \ldots A_n)$

- Combined: $f(x + \lambda y) = f(x) + f(\lambda y)$

  $\det(A_1 \ldots x + \lambda y \ldots A_n) = \det(A_1 \ldots x \ldots A_n) + \lambda \det(A_1 \ldots y \ldots A_n)$

(*)

$$L_{ij}(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & \lambda & & 1 & \\ & & & & & 1 \end{pmatrix} \leftarrow i \qquad I_n = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix} = (e_1, e_2 \dots e_k \dots e_n)$$

$\uparrow j =: k$ $\qquad k$

$$\det L_{ij}(\lambda) = \det(e_1 \dots e_{k-1}, \underbrace{e_k + \lambda e_i}_{k}, e_{k+1} \dots e_n)$$

combined linearity condition

$$= \underbrace{\underbrace{\det(e_1 \dots \underbrace{e_k}_{k} \dots e_n)}_{= I_n}}_{=1} + \lambda \underbrace{\det(e_1 \dots \underbrace{e_i}_{k} \dots e_n)}_{=0} = 1$$

$\square$

<u>Thm. 3.4</u> $A, B \in \mathbb{R}^{n \times n}$. Then $\det AB = \det A \cdot \det B$

For the proof we will need the following <u>observation</u>:

- $P_{ij} \cdot A$ interchanges <u>rows</u> i and j of A

What about $A \cdot P_{ij}$?

Look at $(A P_{ij})^T$:

$$(A \cdot P_{ij})^T = \underbrace{P_{ij}^T}_{= P_{ij}} \cdot A^T = P_{ij} \cdot A^T$$

$$\overset{T}{\Rightarrow} A \cdot P_{ij} = (P_{ij} \cdot A^T)^T$$

Hence : $A P_{ij}$ interchanges <u>columns</u> i and j of A

Analogously:

- $D_i(\lambda) \cdot A$ multiplies row i of A by λ $(\lambda \in \mathbb{R}, \lambda \neq 0)$
  $A \cdot D_i(\lambda)$ multiplies column i of A by λ $(\lambda \in \mathbb{R}, \lambda \neq 0)$

- $L_{ij}(\lambda) \cdot A$ adds λ times j-th row to the i-th row of A
  $A \cdot L_{ij}(\lambda)$ adds λ times i-th column to the j-th column of A

Proof Case 1: B is not invertible $\overset{\text{Lemma 3.2}}{\Rightarrow} \det B = 0$

$\Downarrow$ Exercise 5.

AB is not invertible $\Rightarrow \det(AB) = 0$

$$\Rightarrow \det A \cdot \det B = (\det A) \cdot 0 = \det(AB)$$

Case 2: B is invertible $\Rightarrow B = (B^{-1})^{-1} \Rightarrow B$ is itself an inverse

$\Rightarrow$ We can write B as a product of elementary matrices:

$B = E_1 \cdot \ldots \cdot E_k$

$$\Rightarrow \det(AB) = \det(\underbrace{A E_1 \ldots E_{k-1}}_{= \tilde{A}} E_k)$$

$$= \begin{cases} -\det(A E_1 \ldots E_{k-1}) & \text{if } E_k \text{ is of type 1} \\ \lambda \det(A E_1 \ldots E_{k-1}) & \text{if } E_k \text{ is of type 2} \quad (\lambda \neq 0) \\ 1 \cdot \det(A E_1 \ldots E_{k-1}) & \text{if } E_k \text{ is of type 3} \end{cases}$$

$\leftarrow$ Thm 3.1. (iii)

by $(*)$

Recall Lemma 3.3: $\det(E_k) = \begin{cases} -1 & \text{if } E_k \text{ is of type 1} \\ \lambda & \text{if } E_k \text{ is of type 2} \quad (\lambda \neq 0) \\ 1 & \text{if } E_k \text{ is of type 3} \end{cases}$

$$= \det(AE_1 \dots E_{k-1}) \det(E_k)$$

$= \dots$ proceed inductively

$$= \det(A) \det(E_1) \cdot \ldots \cdot \det(E_k)$$

By the same argument as above

$$\det(B) = \det(\underbrace{E_1 \cdot \ldots \cdot E_k}_{= B}) = \det(E_1) \cdot \ldots \cdot \det(E_k)$$

$$\Rightarrow \det(AB) = \det A \cdot \det B \qquad \square$$

We now know that $\det(A \cdot B) = \det A \cdot \det B$.

Hence: $\det(AB) = \det A \cdot \det B = \det B \det A = \det(BA)$.

(Despite the fact that in general $AB \neq BA$)

Q: $\det(A+B) = \det A + \det(B)$?

No! Because

$$A = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow A + B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\det A = \det B = 1 \qquad \det(A+B) = 4, \qquad 4 \neq 1+1$$

Thm 3.5 Let $A \in \mathbb{R}^{n \times n}$.

(1) A is invertible $\Leftrightarrow$ $\det A \neq 0$

(2) If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det A}$

Proof (1) $\boxed{\Leftarrow}$ Lemma 3.2

(1) $\boxed{\Rightarrow}$ Assume that A is invertible

This means that $AA^{-1} = A^{-1}A = I_n$

$$\Rightarrow \underbrace{\det I_n}_{=1} = \det(AA^{-1}) \underset{\text{Thm 3.4}}{\overset{\uparrow}{=}} \det A \det A^{-1}$$

$\Rightarrow \det A \neq 0$ (otherwise $1 = 0$).

(2) Exercise 6.8 $\square$

Thm 3.6 Let $A \in \mathbb{R}^{n \times n}$. Then, $\det A = \det A^T$

Proof Case 1 : A is not invertible $\overset{\text{Thm 2.7}}{\Rightarrow}$ A^T not invertible

By Thm 3.5 : $\det A = 0$ and $\det A^T = 0$

$\Rightarrow \det A = \det A^T$.

Case 2 $\quad A$ is invertible $\Rightarrow A = E_1 \dots E_k$ product of elem. matrices

$$\Rightarrow A^T = (E_1 \dots E_k)^T = E_k^T \dots E_1^T$$

$$\Rightarrow \det(A^T) = \det(E_k^T \dots E_1^T) \underset{\text{Thm. 3.4}}{=} \det(E_k^T) \dots \det(E_1^T) \quad (*)$$

Looking at the elementary matrices, we observe:

- $L_{ij}(\lambda)^T = L_{ji}(\lambda)$
- $D_i(\lambda)^T = D_i(\lambda)$
- $P_{ij}^T = P_{ij}$

$\Rightarrow \det(E^T) = \det(E)$ for all elementary matrices E.

$$\Rightarrow \det(A^T) \overset{(*)}{=} \det(E_k) \dots \det(E_1)$$

$$= \det(E_1) \dots \det(E_k)$$

can reorder because they are numbers in $\mathbb{R}$.

$$\underset{\text{Thm 3.4}}{=} \det(\underbrace{E_1 \dots E_k}_{=A}) = \det(A). \qquad \square$$

3.3. Computation of determinants

Recall notation: $A \in \mathbb{R}^{n \times n}$ then we write A_{ij} for the matrix in $\mathbb{R}^{(n-1)\times(n-1)}$ that we obtain by deleting the i-th row and the j-th column from A.

Def. Given $A \in \mathbb{R}^{n\times n}$, define $c_{ij} := (-1)^{i+j} \det(A_{ij})$.

We call c_{ij} the (i,j)-cofactor of A.

Observe that by the definition of $\det A$:

$$\det A = a_{11}c_{11} + a_{12}c_{12} + \dots + a_{1n}c_{1n} = \sum_{k=1}^{n} a_{1k}c_{1k}$$

Thm. 3.7 (Laplace expansion)

Let $A \in \mathbb{R}^{n\times n}$ then:

- For every choice of a row i, we can compute the det of A by cofactor expansion along the i-th row:

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + \dots + a_{in}c_{in} = \sum_{k=1}^{n} a_{ik}c_{ik}$$

- For every choice of a column j, we can compute the det of A by cofactor expansion along the j-th column:

$$\det A = a_{1j}c_{1j} + a_{2j}c_{2j} + \dots + a_{nj}c_{nj} = \sum_{k=1}^{n} a_{kj}c_{kj}$$

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ -10 & 0 & 4 \\ 1 & 0 & 0 \end{pmatrix}$

Computing $\det A$ by definition (i.e. expansion along first row)

$$\begin{pmatrix} 1 & 2 & 3 \\ -10 & 0 & 4 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det A = 1 \cdot \underbrace{\det\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}}_{0\cdot 0 - 4\cdot 0 = 0} - 2\det\begin{pmatrix} -10 & 4 \\ 1 & 0 \end{pmatrix} + 3\det\begin{pmatrix} -10 & 0 \\ 1 & 0 \end{pmatrix} = 8$$

$\dots$

Thm 3.7 allows us to pick **any** row$^{\to i}$ or column$^{\to j}$ instead of the first row.

$\rightsquigarrow$ it is smart to pick a row or column with many zeros

e.g. pick 3rd row $(i=3)$:

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det A = 1 \cdot \det \underbrace{\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}}_{2 \cdot 4 - 3 \cdot 0} - 0 + 0 = 8$$

e.g. pick 2nd column $(j=2)$.

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det A = -2 \cdot \det \begin{pmatrix} -1 & 4 \\ 1 & 0 \end{pmatrix} + 0 - 0 = 8$$

$2 = a_{12} \to 1+2$ odd

$\Rightarrow (-1)^{1+2} = -1 \Rightarrow$ start with minus

Q: How do we know whether to start with + or − ?

Expansion along the i-th row:

- starts with $\oplus$ if i odd
- starts with $\ominus$ if i even

Reason: if i odd then $i+1$ even

$\Rightarrow C_{i1} = \underbrace{(-1)^{i+1}}_{=1} \det A_{i1}$ plus!

if i even then $i+1$ odd

$\Rightarrow C_{i1} = \underbrace{(-1)^{i+1}}_{=-1} \det A_{i1}$ minus!

Expansion along the j-th column

- starts with $\oplus$ if j odd
- starts with $\ominus$ if j even

Reason: if j odd then $1+j$ even

$\Rightarrow C_{1j} = \underbrace{(-1)^{1+j}}_{=1} \det A_{1j}$ plus!

if j even then $1+j$ odd

$\Rightarrow C_{1j} = \underbrace{(-1)^{1+j}}_{=-1} \det A_{1j}$ minus!

Proof of Thm 3.7

Part I Computing $\det A$ by expansion along first row (i.e. by def.)

is the same as computing it by expansion along first column, because:

$\det A = \det A^T$ and the first column of A is the first row of A^T.

Part II expansion along j-th column is the same as along first column.

Reason: $A = (A_1 \dots A_j \dots A_n)$

Let $\tilde{A} = (A_j A_1 \dots A_n)$ $\leftarrow$ A rearranged so that the j-th column is first and the rest remains in the usual order.

We can obtain $\tilde{A}$ from A by $j-1$ swaps: $(A_1 A_2 \dots A_{j-1} A_j \dots)$

1. 2. ... j-1

$\underset{\text{Thm 3.1}}{\Rightarrow}$ $\det \tilde{A} = (-1)^{j-1} \det A$ $(*)$

Note, if we can prove that:

the expansion along the j-th column of A
equals $(-1)^{j-1}$ times the expansion along the 1^{st} column of $\tilde{A}$ $\Big]$ $(**)$

then we are done because:

$$(-1)^{j-1} \det A \overset{(*)}{=} \det \tilde{A}$$

$$\overset{\text{Part I}}{=} \text{expansion along } 1^{st} \text{ column of } \tilde{A}$$

$$\overset{(**)}{=} (-1)^{j-1} \left(\text{expansion along the } j^{st} \text{ column of } A\right)$$

$\Rightarrow$ $\det A =$ expansion along the j^{st} column of A

Proof of (**) :

- the expansion along the j-th column of $A = \sum_{k=1}^{n} a_{kj} c_{kj}$

$$= \sum_{k=1}^{n} \underbrace{a_{kj}}_{=\tilde{a}_{k1}} (-1)^{k+j} \det(\underbrace{A_{kj}}_{=\tilde{A}_{k1}}) = \sum_{k=1}^{n} \tilde{a}_{k1} (-1)^{k+j} \det(\tilde{A}_{k1})$$

Same!

- $(-1)^{j-1}$ times the expansion along the 1st column of $\tilde{A}$

$$= (-1)^{j-1} \sum_{k=1}^{n} \tilde{a}_{k1} \tilde{c}_{k1} = (-1)^{j-1} \sum_{k=1}^{n} \tilde{a}_{k1} (-1)^{k+1} \det(\tilde{A}_{k1})$$

Part 3 Row expansion along any row in A equals $\det(A)$.

Reason: Combine Parts I and II with $\det A = \det A^T$. □

Cor. 3.8 Let $A \in \mathbb{R}^{n\times n}$ and let $\tilde{A}$ be the matrix obtained from A by applying one elementary transformation to either rows or columns. Then :

$$\det A = \begin{cases} -\det \tilde{A} & \text{for type 1 (swap)} \\ \frac{1}{\lambda} \det \tilde{A} & \text{for type 2 } (\cdot\lambda) \\ \det \tilde{A} & \text{for type 3 (add multiple)} \end{cases}$$

Proof Exercise

Ex.

type 3

$$\det\begin{pmatrix}1&2&0\\2&4&0\\3&7&1\end{pmatrix} \overset{-2\cdot}{=} \det\begin{pmatrix}1&2&0\\0&0&0\\3&7&1\end{pmatrix}$$

Thm 3.9 Let $A \in \mathbb{R}^{n\times n}$ be an upper or lower triangular matrix. Then $\det A$ is the product of the diagonal entries:

$$\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

Proof: exercise

Thms 3.8 and 3.9 together with row reduction make the computation of determinants much easier because a matrix in echelon form always is triangular.

Ex

Cor 3.8 (type 2)
$\lambda = \frac{1}{2} \Rightarrow \frac{1}{\lambda} = 2$

$$\det\begin{pmatrix}0&2&3&4\\2&2&2&2\\9&10&11&12\\13&14&15&17\end{pmatrix} = 2\det\begin{pmatrix}0&2&3&4\\1&1&1&1\\9&10&11&12\\13&14&15&17\end{pmatrix}$$

$$\overset{\substack{\text{Cor. 3.8}\\ \text{type 1}}}{=} -2 \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 17 \end{pmatrix} \overset{\substack{\text{Cor 3.8}\\ \text{type 3}}}{=} -2 \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{pmatrix}$$

$$\overset{\substack{\text{Cor 3.8}\\ \text{type 3}}}{=} -2 \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\substack{\text{Cor. 3.8}\\ \text{type 1}}}{=} 2 \underbrace{\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\overset{\text{Thm 3.9}}{=} 1\cdot 1\cdot(-1)\cdot 1 = -1}$$

upper triangular

$$= 2\cdot(-1) = \underline{\underline{-2}}$$

Rmk If you compute a determinant, you may use row transformations and/or column transformations.

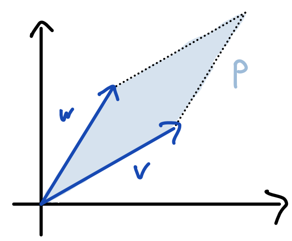
However, if you want to solve a linear system, you can only use row transformations.

(Column transformations would mean confusing the unknowns x_i.)

3.4 Determinants and volume

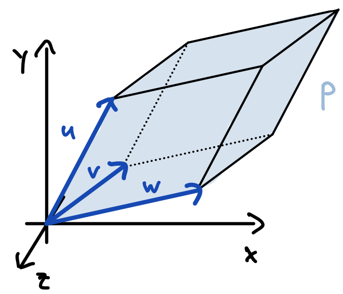
<u>Thm 3.10</u> (i) Let $v, w \in \mathbb{R}^2$ vectors and P the parallelogram given by the two vectors.

Then: $\text{area}(P) = |\det((v, w))|$



(ii) Let $u, v, w \in \mathbb{R}^3$ vectors and P the parallelepiped given by the three vectors.

Then: $\text{Vol}(P) = |\det((u, v, w))|$



<u>Ex.</u> $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \text{area}(P) = ?$

$\text{area}(P) = \left|\det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right| = |1 \cdot 1| = 1$

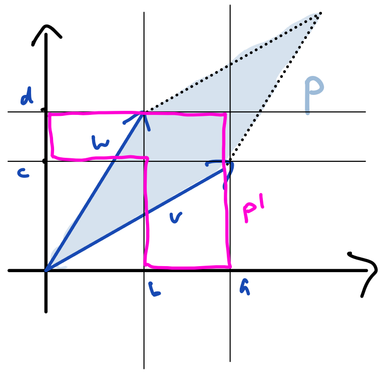
<u>Ex.</u> $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, w = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \text{Area}(P) = \left|\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right| = 0$

$\rightarrow$ What happened ?!

- $\det = 0$ because the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ are collinear.
- But that also means that P is just a line segment and hence has area $= 0$

Formal proof : omitted.

How to semi-formally proof (i) :



$v = \begin{pmatrix} a \\ c \end{pmatrix}$, $w = \begin{pmatrix} b \\ d \end{pmatrix}$

$\det(v,w) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \text{area}(P')$

Cut P into parts and puzzle it together so that it becomes P'

$\Rightarrow \det P = \det P'$

Ex. Consider the points $a = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $c = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, $d = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

a.) Are they the vertices of a parallelogram ?

b.) If yes, compute its area

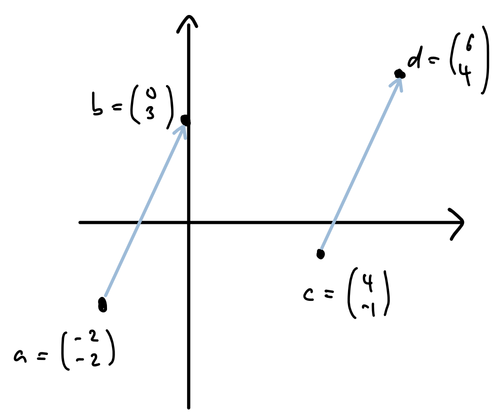
Solution :

a.) It is a parallelogram if and only if there are two pairs of parallel edges.

($\rightsquigarrow$ draw or try all combinations)

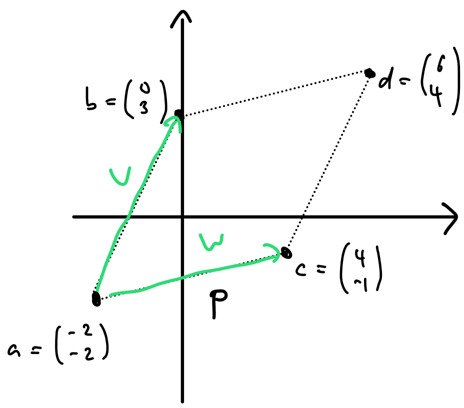
$b - a = \begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$d - c = \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$



$\Rightarrow (a, c, d, b)$ is a parallelogram with parallel sides $b-a$ and $d-c$.

b.) Choose v, w:



$v = b - a = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$w = c - a = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$$\text{area}(P) = |\det(v, w)|$$

$$= \left|\det \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}\right| = |2 - 5 \cdot 6| = 28.$$

4. Vector spaces

Goal: Generalization of $\mathbb{R}^n$ $\rightsquigarrow$ more abstract but also more general concept of "vectors".

Def. A (real) vector space V is a set of objects, that we call vectors, with two operations that we call addition (denoted by the symbol $+$) and scalar multiplication (denoted by $\cdot$) that follow the rules:

$\forall u, v, w \in V, \ \lambda, \mu \in \mathbb{R}$:

(A1) $\quad u + v \in V$

(A2) $\quad u + v = v + u$

(A3) $\quad (u + v) + w = u + (v + w)$

(A4) V contains a zero element :

$\exists\ e \in V$ s.t. $e + u = u \quad \forall u \in V$ "the zero of V"

We will denote this element e by 0_V.

(A5) V contains additive inverses

For every $v \in V$ there exists a $u \in V$ such that $u + v = 0_V$.

We will denote this element u by $-v$.

(S1) $\lambda \cdot u \in V$

(S2) $(\lambda + \mu) u = \lambda u + \mu u$

(S3) $\lambda (u+v) = \lambda u + \lambda v$

(S4) $(\lambda \mu)(v) = \lambda (\mu v)$

(S5) $1 \cdot v = v$

the real number "one"

Examples of vector spaces

1.) $V = \mathbb{R}^n$ with addition and scalar multiplication

2.) $V = \{ f : \mathbb{R}^n \to \mathbb{R}^m \}$ space of all functions $\mathbb{R}^n \to \mathbb{R}^m$

with pointwise addition and pointwise scalar multiplication.

$$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x) \quad \forall x \in \mathbb{R}^n$$

e.g. (A1) is satisfied because indeed : if f and g

are fcts. $\mathbb{R}^n \to \mathbb{R}^m$ then also $f+g$ as defined above is a fct. $\mathbb{R}^n \to \mathbb{R}^m$

(A2) and (A3) are fairly obvious

(A4) is satisfied because if we choose $g_0 : \mathbb{R}^n \to \mathbb{R}^m$ to be the fct. that is constantly equal to $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ (i.e. $g(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \ \forall x \in \mathbb{R}^n$) then:

$$(f+g_0)(x) \underset{\text{def. of } +}{=} f(x) + \underbrace{g_0(x)}_{= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}} = f(x) \quad \forall x \in \mathbb{R}^n \ \forall f : \mathbb{R}^n \to \mathbb{R}^m$$

$\Rightarrow$ g_0 is the zero element of the vector space

(we can write this as: $0_V = g_0$)

3.) Similarly $\tilde{V} = \{ f : \mathbb{R} \to \mathbb{R} \text{ continuous} \}$

and $\tilde{\tilde{V}} = \{ f : \mathbb{R}^n \to \mathbb{R}^m \text{ linear} \}$ are vector spaces. proof later

4.) $V = \{ A \in \mathbb{R}^{m \times n} \}$ is a vector space $\left(\begin{array}{l} + = \text{matrix addition} \\ \cdot = \text{scalar multiplication} \end{array} \right)$

Non-examples

- $V = \{ A \in \mathbb{R}^{n \times n} \mid A \text{ invertible} \}$ is not a vector space.

Let $A = I_n$ and $B = -I_n \Rightarrow A, B \in V$

But $A + B = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \notin V$. $\Rightarrow$ (A1) fails $\Rightarrow$ V is not a vector space.

- $V = \{ f : [0,1] \to [0,1] \}$

(A1) fails: Define $f(x) := 1 \ \forall x \in [0,1]$ and $g(x) := \frac{1}{2} \ \forall x \in [0,1]$
$\Rightarrow f, g \in V$. But $(f+g)(x) = f(x) + g(x) = 1 + \frac{1}{2} = 1.5 \ \forall x \in [0,1]$
$1.5 \notin [0,1] \Rightarrow f+g \notin V \Rightarrow$ (A1) fails $\Rightarrow V$ is not a vector space.

- $V = \{ f : \mathbb{R} \to \mathbb{R} \text{ non-constant} \}$

The only candidate for a zero element in V is the constant zero fct.: $f(x) := 0 \ \forall x \in \mathbb{R}$. But this fct. is not in V.
$\Rightarrow$ (A4) fails $\Rightarrow V$ is not a vector space.

<u>Lemma 4.1</u> Let V be a vector space. Then:

(i) the zero element is unique

(ii) For every $v \in V$: $-v$ is unique

(iii) $0 \cdot u = 0_V \quad \forall u \in V \quad (0 = \lambda \in \mathbb{R})$

(iv) $\lambda \cdot 0_V = 0_V \quad \forall \lambda \in \mathbb{R}$

(v) $-u = (-1) \cdot u \quad \forall u \in V \quad (-1 = \lambda \in \mathbb{R})$

<u>Proof</u> (i) Assume that there are two zero elements 0_V and $\tilde{0}_V$:

So both satisfy: $u + 0_V = u$, $u + \tilde{0}_V = u$ $\quad \forall u \in V$ by (A4)

Choosing $u = \tilde{0}_V$ in first eq. $\Rightarrow \tilde{0}_V + 0_V = \tilde{0}_V$

Choosing $u = 0_V$ in second eq. $\Rightarrow 0_V + \tilde{0}_V = 0_V$

But by (A2): $\tilde{0}_V + 0_V = 0_V + \tilde{0}_V$

$\Rightarrow \tilde{0}_V = 0_V$.

(ii) Let $u \in V$ and assume u has two additive inverses: $-u$ and $\widetilde{-u}$

$$\overset{(A5)}{\Rightarrow} \begin{cases} u + (-u) = 0_V & (*) \\ u + (\widetilde{-u}) = 0_V & (**) \end{cases}$$

$$\Rightarrow -u \overset{(A4)}{=} -u + 0_V \overset{(**)}{=} -u + \left(u + (\widetilde{-u})\right) \overset{(A3)}{=} -u + u + (\widetilde{-u})$$

$$\overset{(A2)}{=} (\widetilde{-u}) + u + (-u) \overset{(A3)}{=} (\widetilde{-u}) + \left(u + (-u)\right) \overset{(*)}{=} (\widetilde{-u}) + 0_V \overset{(A4)}{=} \widetilde{-u}$$

(iii) First: $\underbrace{0}_{\in \mathbb{R}} \cdot u = \underbrace{(0+0)}_{\in \mathbb{R}} u \overset{(S3)}{=} 0 \cdot u + 0 \cdot u$

Now: $0_V \overset{(A5)}{=} 0 \cdot u + (-(0 \cdot u)) = (0 \cdot u + 0 \cdot u) + (-(0 \cdot u))$

$$\overset{(A3)}{=} 0 \cdot u + \left(0 \cdot u + (-(0 \cdot u))\right) \overset{(A5)}{=} 0 \cdot u + 0_V \overset{(A4)}{=} 0 \cdot u$$

(iv) $\lambda \cdot 0_V \overset{(A4)}{=} \lambda \cdot (0_V + 0_V) \overset{(S3)}{=} \lambda 0_V + \lambda 0_V$

Add $-(\lambda \cdot 0_V)$ to both sides $\overset{(A4),(A5)}{\Rightarrow}$ $0_V = \lambda \cdot 0_V$

(v) $u + ((-1)u) \overset{(S5)}{=} 1 \cdot u + ((-1)u) \overset{(S2)}{=} (\underbrace{1 + (-1)}_{=0 \text{ in } \mathbb{R}}) u = 0 \cdot u \overset{(iii)}{=} 0_V$

So by uniqueness of $-u$ (proven in (ii)): $(-1)u = -u$. $\square$

Def. Let V be a vector space. A subset $H \subseteq V$ is called a *subspace* of V if:

(H1) $0_V \in H$

(H2) $u, v \in H \Rightarrow u + v \in H$

(H3) $u \in H, \lambda \in \mathbb{R} \Rightarrow \lambda u \in H$

Lemma 4.2 If H is a subspace of a vector space V then H is itself a vector space.

Proof • (H2) $\Rightarrow$ (A1) for H

• (A2) and (A3) for H follow from the fact that $H \subseteq V$.

• (H1) $\Rightarrow$ (A4) for H

• (A5): Let $v \in H \Rightarrow v \in V \overset{(A5)}{\Rightarrow} \exists -v \in V : v + (-v) = 0_V$

By Lemma 4.1: $-v = \underbrace{(-1)}_{=\lambda \in \mathbb{R}} \underbrace{v}_{\in H} \underset{(H3)}{\Rightarrow} -v \in H$.

• (H3) $\Rightarrow$ (S1) for H

• (S2) until (S5) follow from the fact that $H \subseteq V$ and Lemma 4.1. $\square$

Ex. • $\underbrace{\{ f : \mathbb{R}^n \to \mathbb{R}^m \text{ linear} \}}_{\text{linear fcts.}} \subseteq \underbrace{\{ f : \mathbb{R}^n \to \mathbb{R}^m \}}_{\text{all fcts.}}$ is a subspace

Proof (H1) We know that the constant zero fct. g_0 is the zero element of V. ($0_V = g_0$)

g_0 is linear (given by matrix $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$) $\Rightarrow 0_V \in H$.

(H2) and (H3) are Lemma 1.13. $\square$

- V vector space $\Rightarrow H := \{0_V\}$ is a subspace

  <u>Proof:</u> (H1) is satisfied by def. of H

  (H2): $0_V + 0_V = 0_V \in H$

  (H3): $\lambda \cdot 0_V = 0_V \in H$ $\square$

- Is $H = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$ a subspace? $\to$ <u>No</u>

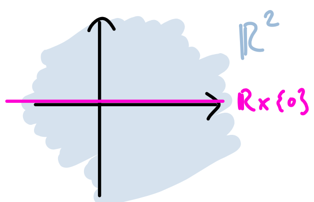
  (H1) is satisfied: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in H$ by definition.

  (H2) first looks like it's ok (e.g. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in H$) but it is not:

  $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \notin H \Rightarrow \text{(H2) fails}$$

  (H3) fails: $3 \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\in H} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \notin H$.

- Is $\mathbb{R} \times \{0\} = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$ is a subspace of $\underset{\mathbb{R}\times\mathbb{R}}{\overset{\shortparallel}{\mathbb{R}^2}}$?



$\to$ <u>Yes!</u> Proof:

<u>(H1)</u> $0_{\mathbb{R}^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R} \times \{0\}$

choose $x = 0$ in $\begin{pmatrix} x \\ 0 \end{pmatrix}$

<u>(H2)</u> Let $u, w \in \mathbb{R} \times \{0\}$

$\Rightarrow u = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} y \\ 0 \end{pmatrix}$ for some $x, y \in \mathbb{R}$

$\Rightarrow u + w = \begin{pmatrix} x+y \\ 0 \end{pmatrix} \in \mathbb{R} \times \{0\}$ (since $x + y \in \mathbb{R}$)

<u>(H3)</u> Let $u \in \mathbb{R} \times \{0\}$, $\lambda \in \mathbb{R}$ $\Rightarrow u = \begin{pmatrix} x \\ 0 \end{pmatrix}$ for some $x \in \mathbb{R}$

$\Rightarrow \lambda u = \begin{pmatrix} \lambda x \\ \lambda \cdot 0 \end{pmatrix} = \begin{pmatrix} \lambda x \\ 0 \end{pmatrix} \in \mathbb{R} \times \{0\}$

(Observe: $\mathbb{R} \times \{0\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$)

Subspaces spanned by vectors

Def. Let V be a vector space and $\{v_1 \dots v_k\} \subseteq V$ a set of vectors.

We define $\quad \text{Span}\{v_1 \dots v_k\} = \left\{ \sum_{i=1}^{k} \lambda_i v_i : \lambda_1, \dots, \lambda_k \in \mathbb{R} \right\}$

We call $\text{Span}\{v_1 \dots v_k\}$ the subspace spanned by (or, generated by) the vectors $v_1, \dots, v_k$.

(**Rmk** One can also define the span of an infinite set W of vectors in V:

$$\text{Span}(W) = \left\{ \sum_{i=1}^{m} \lambda_i w_i : m \in \mathbb{N}, \lambda_1 \dots \lambda_m \in \mathbb{R} \right\}$$

number of vectors m can vary)

not a main focus of this course

Thm 4.3 Let V be a vector space and $\{v_1 \dots v_k\} \subseteq V$ a set of vectors. Then $\text{Span}\{v_1, \dots, v_k\}$ is a subspace of V. More precisely, it is the minimal subspace of V that contains the vectors $v_1, \dots, v_k$. $\to$ proof next week.

Def. Let V be a vector space and $H \subseteq V$ a subspace.

If for some vectors $v_1 \dots v_k \in V$ we have $H = \text{Span}\{v_1, \dots, v_k\}$ then we say that $\{v_1 \dots v_k\}$ is a (finite) generating set for H.

Ex. A plane H in $V = \mathbb{R}^3$ is a subspace (check (H1), (H2), (H3)).

Generating set? Choose any two non-collinear vectors $v, w \in H$

$\Rightarrow H = \operatorname{Span}\{v, w\} \Rightarrow \{v, w\}$ is a generating set.

Q: Does there exist a (finite) generating set for every subspace H of a vector space V?

→ there is no easy "yes/no" answer but we will discuss this question again later.

Q: If a generating set exists, is it unique?

→ No, see e.g. the example above.