Some Theorems that provide practical tools to check for linear independence ...

<u>Thm. 1.7.</u> Let $k \geq 2$ and $\{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$. This set of vectors is linearly dep. if and only if there exists $j \in \{1, \ldots, k\}$ such that v_j is a linear combination of the other vectors.

<u>Proof</u> $\boxed{\Rightarrow}$ Assume $\{v_1, \ldots, v_k\}$ are linearly dep.

We want to show: one of the v_j can be written in terms of all other v_i s.

linear dependence $\Rightarrow$ the equation $\lambda_1 v_1 + \ldots + \lambda_k v_k = 0$ has a non-zero solution

$\Rightarrow$ $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ not all zero, and $\lambda_1 v_1 + \ldots + \lambda_k v_k = 0$

Let $j \in \{1, \ldots, k\}$ so that $\lambda_j \neq 0$.

$$\Rightarrow \quad v_j = -\frac{1}{\lambda_j}\left(\lambda_1 v_1 + \ldots \lambda_{j-1} v_{j-1} + \lambda_{j+1} v_{j+1} + \ldots + \lambda_k v_k\right)$$

$$= \underbrace{-\frac{\lambda_1}{\lambda_j} v_1 - \ldots - \frac{\lambda_{j-1}}{\lambda_j} v_{j-1} - \frac{\lambda_{j+1}}{\lambda_j} v_{j+1} \ldots - \frac{\lambda_k}{\lambda_j} v_k}_{\text{this is a linear comb. of } v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_k \ \checkmark}$$

$\boxed{\Leftarrow}$ Similar (slightly easier) → Exercises $\square$

<u>Thm. 1.8</u> • If $k > n$, then any set $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is linearly dependent.

• If $v_j = 0$ for some index $j \in \{1, \dots, k\}$ then $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is lin. dep.

<u>Proof</u>: → Exercises

<u>Thm. 1.9</u> Let $v_1, \dots, v_{k+1} \in \mathbb{R}^n$ vectors

• If $\{v_1, \dots, v_k\}$ are lin. dep., then also $\{v_1, \dots, v_k, v_{k+1}\}$ are dependent.

• If $\{v_1, \dots, v_{k+1}\}$ are lin indep., then also $\{v_1, \dots, v_k\}$ are lin indep.

<u>Proof</u>: First part: Assume $\{v_1, \dots, v_k\}$ are lin. dep.

$\Rightarrow$ $\lambda_1 v_1 + \dots + \lambda_k v_k = 0$ has a non-zero solution: $\lambda_1, \dots, \lambda_n$ and $\lambda_j \neq 0$ for some j

Cor. 1.10 $A \in \mathbb{R}^{m \times n}$ with columns $A_1, \dots, A_n \in \mathbb{R}^m$.

Consider the equation $Ax = b$, $b \in \mathbb{R}^n$.

(i) for every $b \in \mathbb{R}^m$, the equation $Ax = b$ has at most one solution

$\iff$ $\{A_1, \dots, A_n\}$ linearly independent.

(ii) for every $b \in \mathbb{R}^m$, the equation $Ax = b$ has at least one solution

$\iff$ $\mathrm{Span}\{A_1, \dots, A_n\} = \mathbb{R}^m$

(iii) for every $b \in \mathbb{R}^m$, the equation $Ax = b$ has $\overbrace{\text{a unique solution}}^{\text{i.e. exactly one solution}}$

$\iff$ $\{A_1, \dots, A_n\}$ linearly independent and $\mathrm{Span}\{A_1, \dots, A_n\} = \mathbb{R}^m$

Proof: → exercises

1.7. Matrices and linear maps (1.8 and 1.9 in the book)

We already know: every matrix $A \in \mathbb{R}^{m \times n}$ corresponds to a (family of) system(s) of linear equations $Ax = b$ $(b \in \mathbb{R}^m)$

Now we will learn: it also corresponds to a linear function:

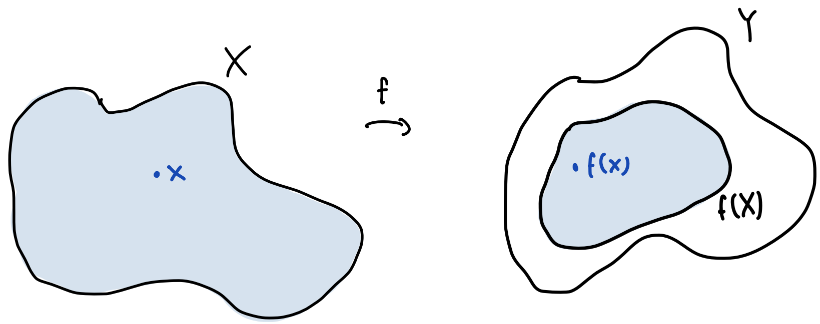
$$\underset{\in \mathbb{R}^n}{x} \longmapsto \underset{\in \mathbb{R}^m}{Ax}, \quad \text{i.e. } f(x) = Ax$$

But let us first learn about functions and, specifically, linear functions...

Def. Let X and Y be two non-empty sets. A function (or map) f is a rule that assigns to each element $x \in X$ a unique element $f(x) \in Y$. We write: $f: X \to Y$

X is called the domain of f and Y is called the codomain (or target space) of f.

The range (or image) of f is: $\{f(x) : x \in X\} =: f(X)$



Ex.

- $X = \mathbb{R},\ Y = \mathbb{R},\ f: \mathbb{R} \to \mathbb{R},\ f(x) = \cos(x)$

  $f(X) = [-1,1] \neq Y$

- $X = \{a, b, c, d\},\ Y = \{*, \heartsuit\},\ f(x) = \begin{cases} * & \text{if } x = a, b \\ \heartsuit & \text{if } x = c, d \end{cases}$

- $X = \mathbb{Z},\ Y = \{0, 1\},\ f(x) = \begin{cases} 0 & \text{if } x \text{ even} \\ 1 & \text{if } x \text{ odd} \end{cases}$

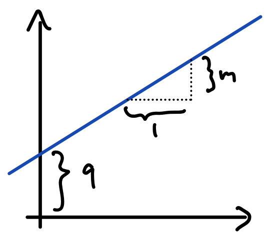
- $X = \mathbb{R}^n,\ Y = \mathbb{R}^m,\ f(x) = Ax$ where $A \in \mathbb{R}^{m \times n}$ some given matrix

Example (from high school):

$X = Y = \mathbb{R}$, $m, q \in \mathbb{R}$ constants

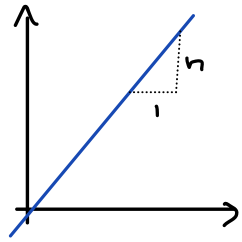
$$f(x) = mx + q$$

$f(x) = mx + q$



affine fct.

$f(x) = mx$ $(q = 0)$



linear fct.

meaning f is linear

We observe: If $q = 0$, then

- $f(x+y) = m(x+y) = mx + my = f(x) + f(y)$
- $f(\lambda x) = m(\lambda x) = \lambda(mx) = \lambda f(x)$ (for $\lambda \in \mathbb{R}$)

This observation motivates the following definition:

Def. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called linear if for all $x, y \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$:

(i) $f(x+y) = f(x) + f(y)$

(ii) $f(\lambda x) = \lambda f(x)$

We observe:

- Linearity (i.e. conditions (i) and (ii) together) are equivalent to the condition:

$$f(\lambda x + y) = \lambda f(x) + f(y) \qquad (\to \text{exercises})$$

- By Prop. 1.5: the map $f(x) = Ax$ is linear $\leftarrow$

Q: Are there linear maps $f: \mathbb{R}^n \to \mathbb{R}^m$ other than maps given by matrix multiplication?

$\to$ No! All linear maps $f: \mathbb{R}^n \to \mathbb{R}^m$ are of the form $f(x) = Ax$ for some matrix $A \in \mathbb{R}^{m \times n}$ ($\to$ proof : later)

Ex. Q: Is $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ x-y \end{pmatrix}$ linear?

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = f\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} \neq$$

$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} + f\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$\Rightarrow$ f does not satisfy cond (i). Hence f is not linear.

Q: Can f be written as $f(x) = Ax$ for some matrix A?

$\to$ No! Because fcts. of the type $h(x) = Ax$ are linear and f is not linear.

Ex. Q: Is $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(v) = 2v$ linear?

$$f(v+w) = 2(v+w) = 2v + 2w = f(v) + f(w) \quad \text{(i) } \checkmark$$

$$f(\lambda v) = 2(\lambda v) = \lambda(2v) = \lambda f(v) \quad \text{(ii) } \checkmark$$

$\Rightarrow$ f is linear.

Q: Can f be written as $f(x) = Ax$ for some matrix A ?

$\to$ Yes ! Define $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ then indeed we have

$$Av = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2v = f(v) \quad \checkmark$$

This gives an alternative solution for the first question:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow f(x) = Ax \text{ for all } x \in \mathbb{R}^3$$

where $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\Rightarrow$ f linear

⚠ If you check for linearity :

- To show that a map is linear :
  - either show the conditions are met by all (!) x, y in the domain ($\to$ work with variables)
  - or find a matrix A so that $f(x) = Ax$ for all x.

- To show that a map is <u>not</u> linear :

  find specific choices of x and y (and λ) so that at least one condition fails

  ($\to$ work with concrete numbers)

Important properties of linear maps

Q: Assume $f: \mathbb{R}^n \to \mathbb{R}^m$ linear. What do we know about $f(0)$?

$$\to \quad f(\underbrace{0}_{\text{vector}}) = f(\underbrace{0}_{\text{scalar}} \cdot \underbrace{0}_{\text{vector}}) \overset{(ii)}{=} 0 \cdot f(0) = 0 \quad \Rightarrow f(0) = 0.$$

Ex. Is $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$ a linear map?

$\to$ No! Because $\cos(0) \neq 0$

Is $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ a linear map?

$\to$ No! Even though $\sin(0) = 0$, $f(x) = \sin(x)$ is not linear

because e.g. $\sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin(\pi) = 0$

$$\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1 + 1 = 2 \neq 0$$

We learn: $f(0) = 0$ is a property of linear maps that is a useful necessary criterion but not a sufficient criterion for linearity

if $f(0) \neq 0$ then f is not linear

if $f(0) = 0$ then f does not need to be linear.

Lemma 1.11 Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map then: (i) $f(0) = 0$

(ii) For a linear combination $\lambda_1 v_1 + \ldots + \lambda_k v_k$ with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, $v_1, \ldots, v_k \in \mathbb{R}^n$, we have:

$$f(\underbrace{\lambda_1 v_1 + \ldots + \lambda_k v_k}_{\text{linear combination of } v_1, \ldots, v_k}) = \underbrace{\lambda_1 f(v_1) + \ldots + \lambda_k f(v_k)}_{\text{linear combination of } f(v_1), \ldots, f(v_k)}$$

Proof: (i) see above

(ii) Apply properties (i) and (ii) from the definition of linearity

Examples of linear maps $f: \mathbb{R}^2 \to \mathbb{R}^2$

(1.) Stretching by a factor

For a constant $\alpha \in [0, \infty)$: $f(x) = \alpha x$

$$f(\lambda x + y) = \alpha(\lambda x + y) = \alpha \lambda x + \alpha y = \lambda \alpha x + \alpha y$$

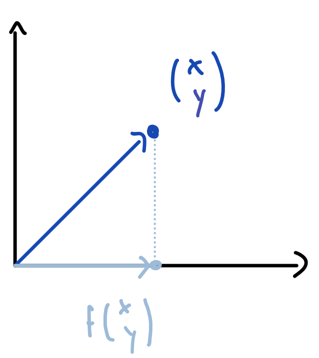
$$= \lambda f(x) + f(y) \Rightarrow f \text{ linear}$$

And we can write f as $f(x) = Ax$ for $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$

(2) Projection onto coordinate axis

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

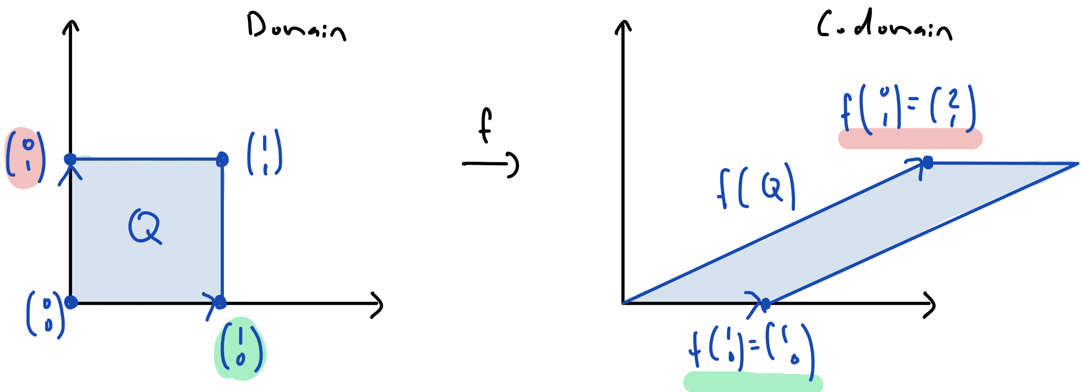


In matrix form:

$$\underbrace{\begin{pmatrix} x \\ 0 \end{pmatrix}}_{f\begin{pmatrix} x \\ y \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

(3.) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix}x\\y\end{pmatrix} := \underbrace{\begin{pmatrix}1&2\\0&1\end{pmatrix}}_{=A}\begin{pmatrix}x\\y\end{pmatrix}$

Geometry :



$$Q = \{x \in \mathbb{R}^2 : 0 \le x_1 \le 1 ,\ 0 \le x_2 \le 1\}$$

$$= \left\{ \lambda_1 \begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\end{pmatrix} : \lambda_1, \lambda_2 \in [0,1] \right\}$$

$$\left.\begin{array}{l} \text{We know: } f\left(\lambda_1 \begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\end{pmatrix}\right) = \lambda_1 f\begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 f\begin{pmatrix}0\\1\end{pmatrix} \\ f\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \qquad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \end{array}\right\} (*)$$

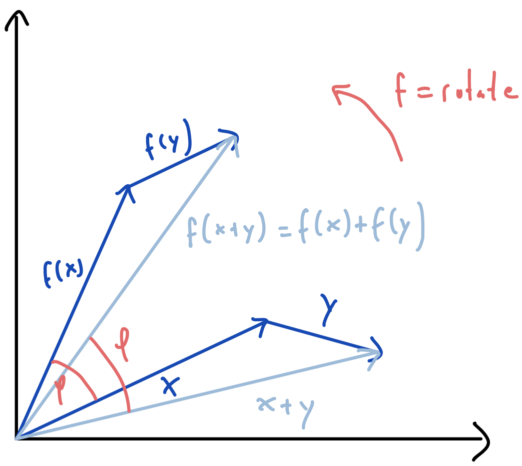
$$\Rightarrow f(Q) \underset{(*)}{=} \left\{ \lambda_1 \begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\end{pmatrix} : \lambda_1, \lambda_2 \in (0,1) \right\}$$

(4.) Rotations in $\mathbb{R}^2$

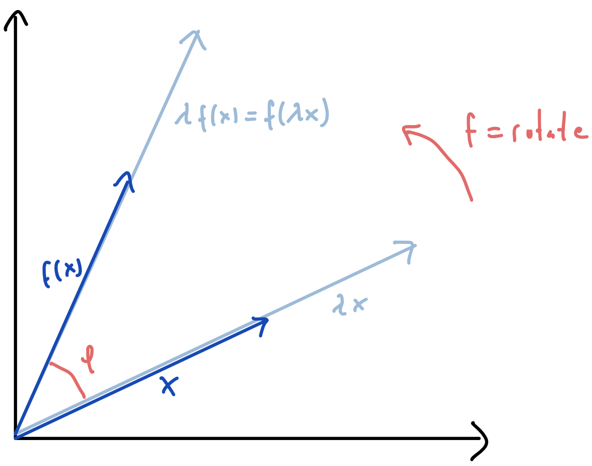
Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be the map that rotates every vector x by a fixed angle $\varphi \in [0, 2\pi)$ counter-clockwise.

Q: Is f linear? → Yes! Because...

Intuitively :



Given two vectors, adding them and then rotating the sum has the same outcome as first rotating each of them then adding them ($\Rightarrow$ condition (i) ✓.)

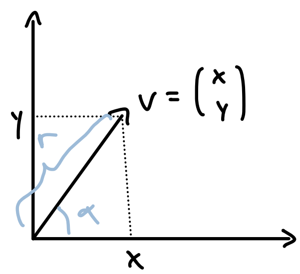


We can argue similarly for condition (ii) i.e. construct!

Formal argument: Find a matrix $A \in \mathbb{R}^{2\times 2}$ so that

$$f(v) = Av.$$

Write the vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in polar coordinates:



$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} = r\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Then $\quad f(v) = r\begin{pmatrix} \cos(\alpha+\varphi) \\ \sin(\alpha+\varphi) \end{pmatrix}$ $\quad\leftarrow$ Want to rewrite this as $A\cdot\underbrace{\begin{pmatrix} r\cos(\alpha) \\ r\sin(\alpha) \end{pmatrix}}_{=x}$

$$= r\cdot\begin{pmatrix} \cos\varphi\cos\alpha - \sin\varphi\sin\alpha \\ \sin\varphi\cos\alpha + \cos\varphi\sin\alpha \end{pmatrix}$$

by additive laws of trigonometric fcts.

$$= r\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\cdot\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

by matrix multipl. rule

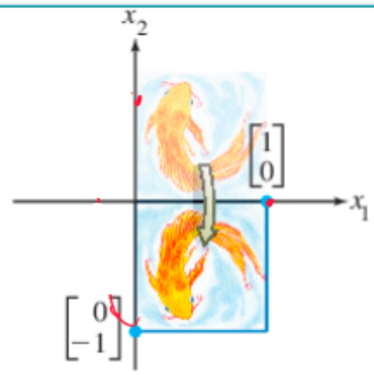
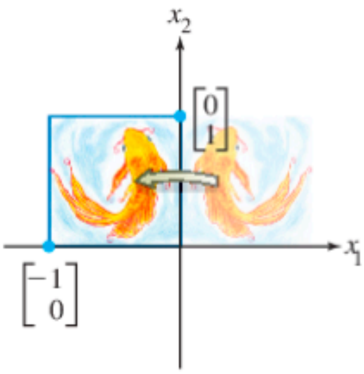
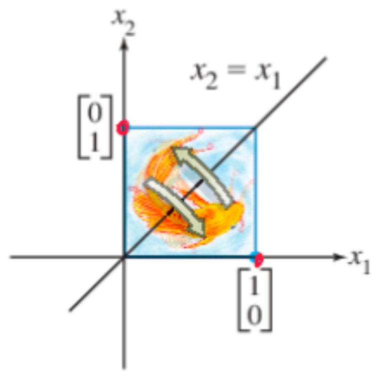
$$= \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{=:A}\cdot\underbrace{\left(r\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\right)}_{=v}$$

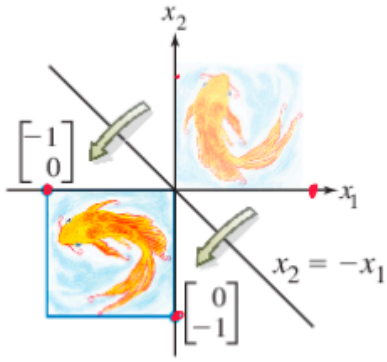
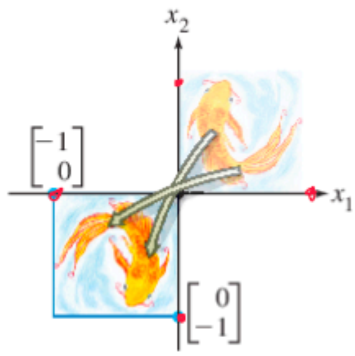
rotation matrix

More examples

Above we saw 3 types of linear maps $f: \mathbb{R}^2 \to \mathbb{R}^2$: stretch by a factor, rotation, projection onto axis. Here are further examples.

As above, we illustrate the geometry of a linear map by studying the image of a square, by determining the image of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$
Reflection through the origin		$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

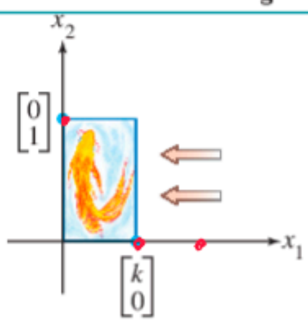
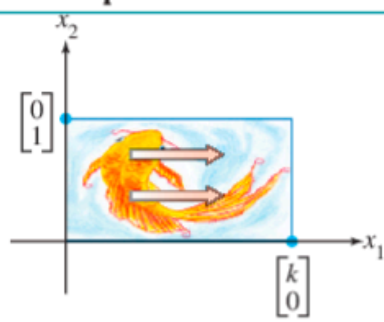
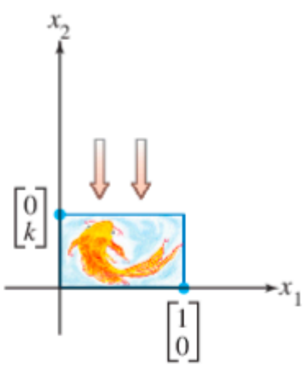
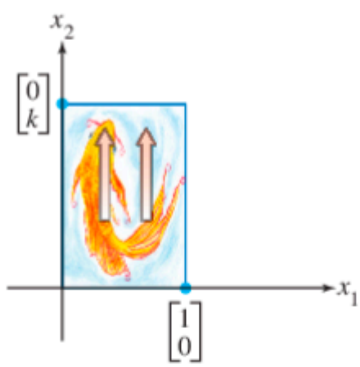
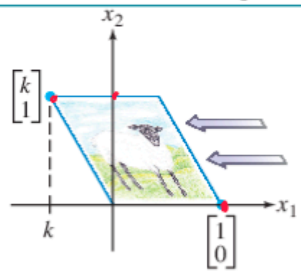
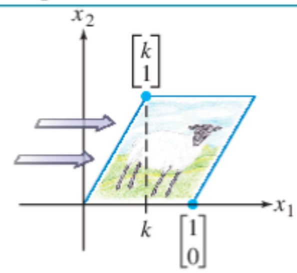
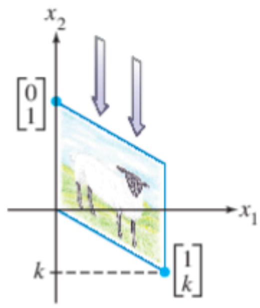
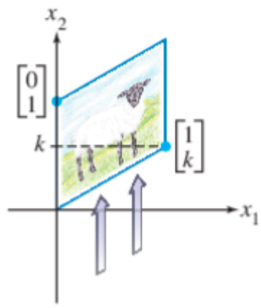
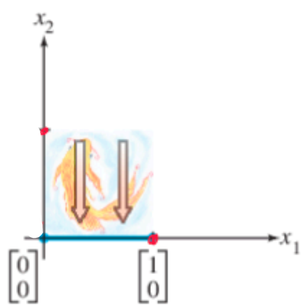
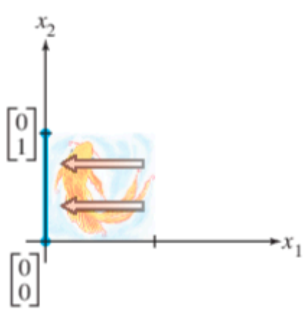
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$

TABLE 3 Shears

Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Thm. 1.12 (i) For every matrix $A \in \mathbb{R}^{m \times n}$, the function

$f_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f_A(x) := Ax$ is linear

(ii) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then there exists

a unique matrix $A \in \mathbb{R}^{m \times n}$ so that f can be

written as $f(x) = Ax$ for all $x \in \mathbb{R}^n$.

This matrix is unique and given by $A = (f(e_1), \dots, f(e_n))$.

Proof: (i) We already know (Prop. 1.5)

(ii) We write $x \in \mathbb{R}^n$ as a linear combination:

(Compare to Example (4) above!)

$x = x_1 e_1 + \dots + x_n e_n$ (see week 2)

By Lemma 1.11: $f(x) = x_1 f(e_1) + \dots + x_n f(e_n)$

By first definition of vector-matrix-multiplication:

$$f(x) = \underbrace{(f(e_1) \dots f(e_n))}_{\text{columns}} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$\Rightarrow$ We found A: $A = (f(e_1) \dots f(e_n))$

Uniqueness: We have to show that whenever $f(x) = Bx$

for all $x \in \mathbb{R}^n$ then B must equal A.

Assume $f(x) = Bx$ for all $x \in \mathbb{R}^n$

the i-th column of B

Plug in e_i for x: $f(e_i) = B \cdot e_i = B_i$

However, we also have $f(e_i) = A e_i = A_i$

$\Rightarrow B_i = f(e_i) = A_i$ (i-th columns are the same!)

We can do this for all columns. $\Rightarrow A = B$. $\square$

__Ex.__ $f: \mathbb{R}^4 \to \mathbb{R}^4$, $f(x_1, x_2, x_3, x_4) = \begin{pmatrix} 2x_1 + 3x_2 - x_4 \\ -2x_3 \\ x_1 + x_4 \\ -\sqrt{2}x_1 + x_3 \end{pmatrix}$

Q: Write f in matrix form

$$f(e_1) = \begin{pmatrix} 2 \\ 0 \\ 1 \\ -\sqrt{2} \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f(e_3) = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \quad f(e_4) = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Thm. $\Rightarrow$ $A = \begin{pmatrix} 2 & 3 & 0 & -1 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 1 \\ -\sqrt{2} & 0 & 1 & 0 \end{pmatrix}$

The columns of A are the vectors $f(e_1), f(e_2), f(e_3), f(e_4)$

Same but faster:

$$f(x_1, x_2, x_3, x_4) = \begin{pmatrix} 2x_1 + 3x_2 - x_4 \\ -2x_3 \\ x_1 + x_4 \\ -\sqrt{2}x_1 + x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 & +3x_2 & 0 & -1x_4 \\ 0 & 0 & -2x_3 & 0 \\ 1x_1 & 0 & 0 & +1x_4 \\ -\sqrt{2}x_1 & 0 & +1x_3 & 0 \end{pmatrix}$$

Write parts for each x_i nicely under each other.

columns of A.

Adding and rescaling functions

Def. Given two functions $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^n \to \mathbb{R}^m$ and a scalar $\lambda \in \mathbb{R}$, we define the functions $f+g$ and λf as follows:

- $(f+g)(x) = f(x) + g(x)$ for all $x \in \mathbb{R}^n$
- $(\lambda f)(x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$

Example $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x$

$\Rightarrow \underbrace{f+g}_{=:h} : \mathbb{R} \to \mathbb{R}$, $\underbrace{(f+g)(x)}_{h(x)} = \cos(x) + 2x$

Lemma 1.13 If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^n \to \mathbb{R}^m$ are linear fcts. and $\lambda \in \mathbb{R}$ a scalar, then:

$f+g$ as well as λf are also linear fcts. $\mathbb{R}^n \to \mathbb{R}^m$

Proof :

$$(f+g)(x+y) \overset{\text{Def. of } (f+g)}{=} f(x+y) + g(x+y) \overset{\text{linearity of } f \text{ and } g}{=} f(x) + f(y) + g(x) + g(y)$$

$$\underset{\text{reorder}}{=} f(x) + g(x) + f(y) + g(y) \underset{\text{Def. of } (f+g)}{=} (f+g)(x) + (f+g)(y)$$

$\Rightarrow$ $f+g$ satisfies property (i) of Def. of linearity

Similarly : property (ii) for $f+g$, and properties (i) and (ii) for λf. $\square$

Alternative proof :

f and g are linear

$\Rightarrow$ There exist matrices A and B so that :
Thm 1.12 $f(x) = Ax$ and $g(x) = Bx$ for all $x \in \mathbb{R}^n$

$\Rightarrow$ $(f+g)(x) = f(x) + g(x) = Ax + Bx = \underbrace{(A+B)}x$

Sum of matrices. Formal definition in Section 2 (later)

$\Rightarrow$ $f+g$ is given by a matrix

$\Rightarrow$ $f+g$ linear.

Analogously : $(\lambda f)(x) = (\lambda A)x$ $\square$

Injective, surjective, bijective

Def. We say that a fct. $f: X \to Y$ is

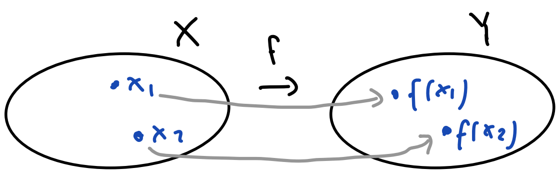
- injective if: whenever $f(x_1) = f(x_2)$ then it follows that $x_1 = x_2$
- surjective if: for all $y \in Y$ there exists $x \in X$ s.t. that $f(x) = y$.
- bijective if it is both injective and surjective

Synonyms:

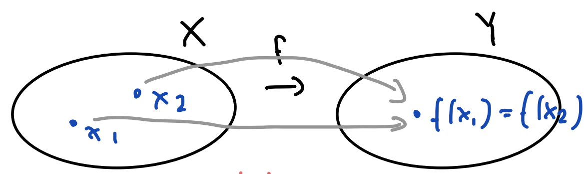
- injective = one-to-one
- surjective = onto

How to think about these notions:

Injectivity:



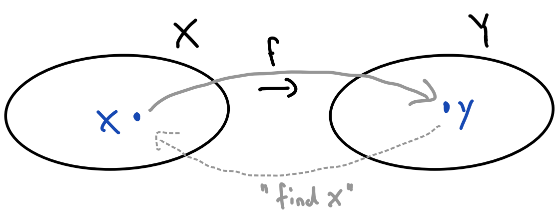
injective



Not injective

injective means: "two distinct pts cannot map to the same pt."

Surjectivity



surjective means "the codomain is the range of the map" $(Y = f(X))$

Ex.

- $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
  - injective: no! Because $f(2) = 4 = f(-2)$ but $2 \neq -2$
  - surjective: no! There is no $x \in \mathbb{R}$ so that $f(x) = -4 \in \mathbb{R}$

- $f: [0,\infty) \to [0,\infty)$, $f(x) = x^2$
  - injective: $\Big\}$ both yes! Because:
  - surjective: each $y \in [0,\infty)$ $\underbrace{\text{has}}_{\text{surj.}}$ a $\underbrace{\text{unique}}_{\text{inj}}$ square root $x = \sqrt{y} \in [0,\infty)$

- $f: [0,\infty) \to \mathbb{R}$, $f(x) = x^2$
  - injective: yes! If $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2 \Rightarrow x_1 = x_2$ ($x_1, x_2 \in [0,\infty)$)
  - surjective: no! Same as first example above.

- $f: \mathbb{R}^2 \to \mathbb{R}$, $f\begin{pmatrix} x \\ y \end{pmatrix} = x - y^2$
  - injective: no! Because $f\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 = f\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ but $\begin{pmatrix} -1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
  - surjective: yes! Let $r \in \mathbb{R}$, choose $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$, then: $f\begin{pmatrix} r \\ 0 \end{pmatrix} = r$

- $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \arctan\left(\log|x| + e^{x^3} - x^{17} + \sin x\right)$
  - injective: ?? Who knows... 🤷‍♀️
  - surjective: ??

$\rightsquigarrow$ It is in general very hard to say which fcts are inj., surj., or bij.

Even for $\mathbb{R} \to \mathbb{R}$ it is hard. Let alone $\mathbb{R}^n \to \mathbb{R}^m$

For linear fcts, it is actually not hard at all to find out whether they are injective, surjective, and/or bijective.

Let us first give a simpler characterization of injectivity and surjectivity for the case of linear fcts.

Thm. 1.14 Consider a linear map $f: \mathbb{R}^n \to \mathbb{R}^m$ given by a matrix $A \in \mathbb{R}^{m \times n}$, that is, $f(x) = Ax$. Then:

(i) f is injective $\Leftrightarrow$ whenever $f(x) = 0$ then $x = 0$

(i.e. $x = 0$ is the only point where $f(x) = 0$)

(ii) f is surjective $\Leftrightarrow$ $\operatorname{Span}\{A_1, \dots, A_n\} = \mathbb{R}^m$

Proof (i) $\boxed{\Rightarrow}$ Assume f is injective. Assume $f(x) = 0$.

(We want to deduce that $x = 0$.)

Since f linear: $f(0) = 0$ $\Rightarrow f(x) = f(0) \overset{f \text{ injective}}{\Rightarrow} x = 0$

$\boxed{\Leftarrow}$ Now assume that: whenever $\underbrace{f(x) = 0 \text{ then } x = 0}_{=: *}$

Assume that $f(x_1) = f(x_2)$. (We want to deduce that $x_1 = x_2$)

$$f(x_1) = f(x_2) \Rightarrow 0 = f(x_1) - f(x_2) \underset{f \text{ linear}}{=} f(x_1 - x_2)$$

$$\underset{* \text{ with } x = x_1 - x_2}{\Rightarrow} x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

(ii) f surjective $\overset{\text{def.}}{\iff}$ for every $y \in \mathbb{R}^m$ there exists $x \in \mathbb{R}^n$ such that $\underbrace{f(x) = y}$ $\iff Ax = y$

$\iff$ for every $y \in \mathbb{R}^m$: $Ax = y$ has a solution

$\underset{\text{Thm. 1.4}}{\iff}$ $\text{Span}\{A_1, \dots, A_n\} = \mathbb{R}^m$. $\square$

<u>Cor 1.15</u> Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear fct.

(i) if $m > n$ then f cannot be surjective

(ii) if $m < n$ then f cannot be injective

(iii) if $m \neq n$ then f cannot be bijective

(a pivot of a matrix = a leading entry of a row of the matrix)

<u>Proof</u> (i) Assume $m > n$

An echelon form of A can have at most n pivot elements $\Rightarrow$ not every row has a pivot element

$\uparrow$ number of columns

$\underset{\text{Thm. 1.4}}{\Rightarrow}$ $\text{Span}\{A_1 \dots A_n\} \neq \mathbb{R}$ $\underset{\text{Thm 1.14}}{\Rightarrow}$ f not surjective

(ii) Assume $m < n$ $\iff$ The matrix has less rows than columns

$\Rightarrow$ every echelon form has free variables

$\Rightarrow$ $Ax = 0$ has several solutions $\underset{\text{Thm. 1.4}}{\Rightarrow}$ f is not injective.

(iii) Follows from (i) and (ii). $\square$

Thm. 1.16 Consider a linear map $f: \mathbb{R}^n \to \mathbb{R}^m$ given by a matrix $A \in \mathbb{R}^{m \times n}$, that is, $f(x) = Ax$. Let $\tilde{A}$ be an echelon form of A. Then:

(i) f is injective $\iff$ $\tilde{A}$ has n pivot elements

(ii) f is surjective $\iff$ $\tilde{A}$ has m pivot elements

(iii) f is bijective $\iff$ $n = m$ and $\tilde{A}$ has n pivot elements

Proof (i) $\tilde{A}$ has n pivot elements

$\iff$ every column has a pivot element

$\iff$ no free variables

$\iff$ $Ax = 0$ has a unique solution

$\iff$ $x = 0$ is the only x for which $f(x) = 0$

$\underset{\text{Thm 1.14}}{\iff}$ f is injective

(ii) $\tilde{A}$ has m pivot elements

$\iff$ every row has a pivot elemen

$\iff$ no zero rows

$\iff$ $Ax = b$ has a solution for every $b \in \mathbb{R}^m$

$\iff$ for every $b \in \mathbb{R}^m$ there exists an $x \in \mathbb{R}^n$ such that $f(x) = b$

$\underset{\text{Def.}}{\iff}$ f surjective

(iii) Is a direct consequence of (i) and (ii).

Additional material

Here are a few concrete examples of linear maps:

a.) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = Ax$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

- A is in echelon form and has a pivot element in every column (i.e. has n pivot elements)

  $\underset{\text{Thm. 1.16}}{\Longrightarrow}$ f is injective

  Or if you want to go by definition:

  Assume $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

  This means $\begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ 2y_2 \end{pmatrix}$

  $\Rightarrow x_1 = y_1$ and $\underbrace{2x_2 = 2y_2}_{\Rightarrow x_2 = y_2}$ $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

- A is in echelon form and has a pivot in every row (i.e. has m pivot elements)

  $\underset{\text{Thm. 1.16}}{\Longrightarrow}$ f is surjective

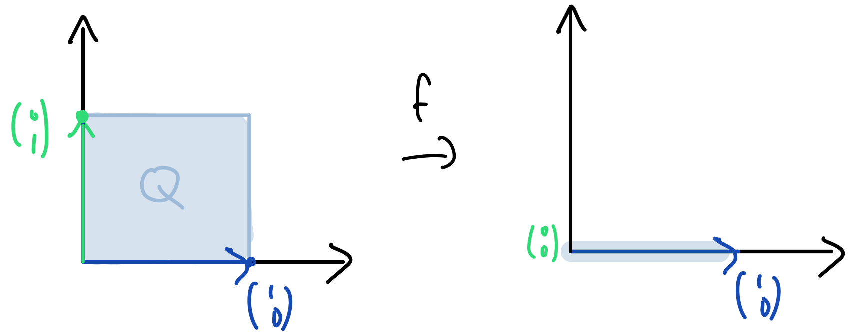
- f is inj. and surj. so it is also bijective.

Additional material

b.) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = Ax$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

- f is not injective : $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- In fact $A\begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ regardless of y

$\Rightarrow$



$f(Q)$ is just the interval $[0,1]$ on the x-axis

- f is not surjective :

For $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $Ax = b$ has no solution.

Why? Either argue by Thm 1.16 or use augmented matrix.

c.) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = Ax$, $A = \begin{pmatrix} 1 & 0 \end{pmatrix}$

- Not injective (see b.))
- surjective : For every $b \in \mathbb{R}$: $A\overbrace{\begin{pmatrix} b \\ 0 \end{pmatrix}}^{=x} = b$

Next week Tuesday at ~14:30-15:00 :

Graded exercise I

- Two short exam-style problems to hand-in at 15:00.
- Bring pen and paper !!!
- Topics to prepare :
    - systems of linear equations
      matrix-vector equations ($Ax=b$)
      solution spaces
    - linear combinations of vectors
      linear dependence and independence
      span