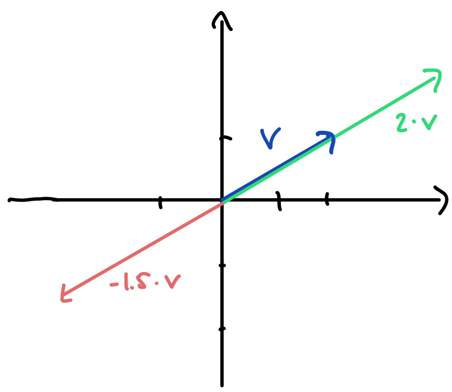
Geometric interpretation of vector arithmetic

$v, w \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

$\lambda \cdot v \;\hat{=}$ stretching v by a factor λ

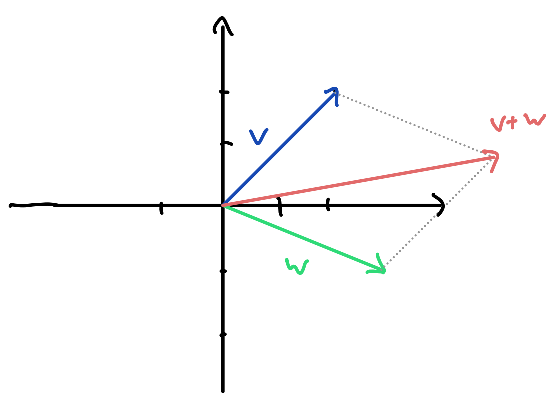
(if $\lambda < 0$: stretch by $|\lambda|$ and revert direction)

<u>Ex.</u> $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$



$v + w \;\hat{=}$ the forth corner of the parallelogram with corners 0, v, and w.

<u>Ex.</u> $v = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix} \in \mathbb{R}^2$



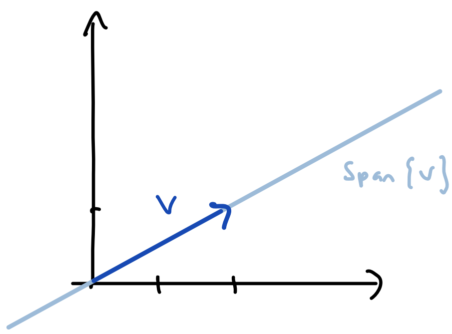
Geometric interpretation of span

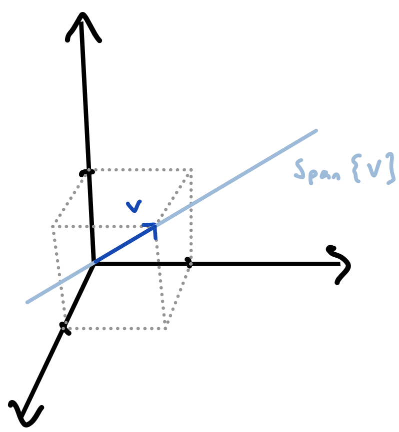
The span of one vector is a line (in $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^n$, $n>3$) in the direction of the vector.

e.g. $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$

$$\text{Span}\{v\} = \left\{ \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}$$

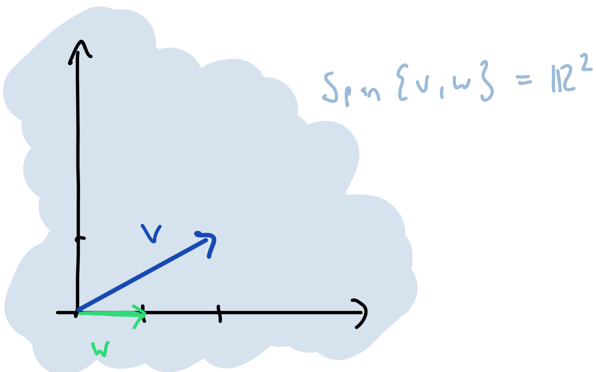
$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$





The span of two vectors (unless they are a scalar multiple of each other) is a plane spanned by their directions:

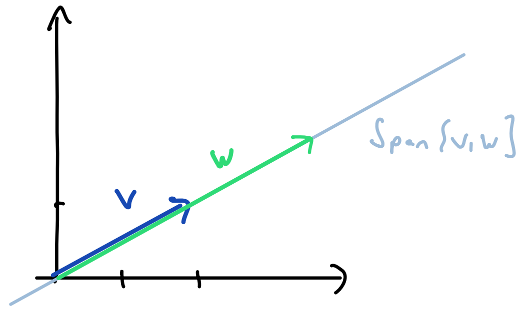
$v = \begin{pmatrix}2\\1\end{pmatrix}, \; w = \begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$



Span$\{v,w\}$ is a plane because

$v \neq \lambda w \quad \forall \lambda \in \mathbb{R}$

$v = \begin{pmatrix}2\\1\end{pmatrix}, \; w = \begin{pmatrix}4\\2\end{pmatrix}$

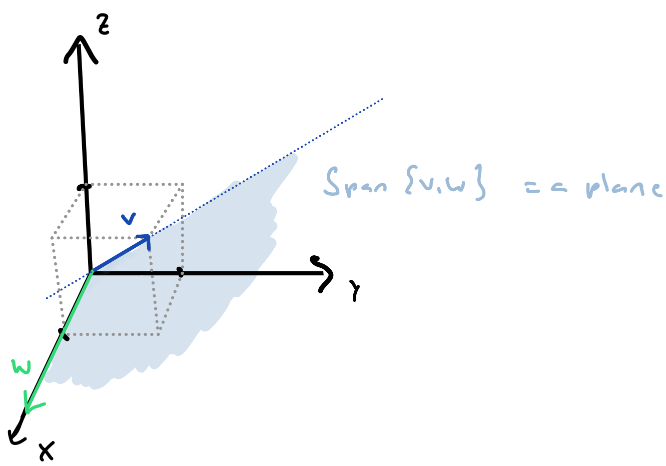


Span$\{v,w\}$ is a line because

$w = 2 \cdot v$

$v = \begin{pmatrix}1\\1\\1\end{pmatrix}, \; w = \begin{pmatrix}2\\0\\0\end{pmatrix} \in \mathbb{R}^3$

$(v \neq \lambda w, \; \lambda \in \mathbb{R})$



1.4. The matrix equation

Def. (Ax by adding up columns)

Let A be an $m \times n$-matrix with columns $A_1, \dots, A_n \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. The product $A \cdot x$ (or short: Ax) is defined by:

$$Ax := \underbrace{\overset{\in\mathbb{R}}{x_1}\,\overset{\in\mathbb{R}^m}{A_1}}_{\in\mathbb{R}^m} + \dots + x_n A_n \qquad \in \mathbb{R}^m$$

Note: This only makes sense if # columns of A = # entries of x.

Ex.

$$\begin{array}{c} \begin{pmatrix}1&2&1\\0&1&-1\end{pmatrix} \\ \begin{matrix}A_1&A_2&A_3\end{matrix}\end{array} \begin{pmatrix}2\\1\\3\end{pmatrix}\begin{array}{l}x_1\\x_2\\x_3\end{array} = 2\begin{pmatrix}1\\0\end{pmatrix}+1\cdot\begin{pmatrix}2\\1\end{pmatrix}+3\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}7\\-2\end{pmatrix}$$

$(n = 3,\ m = 2)$

Alternative way to compute the product:

$$\begin{pmatrix}1&2&1\\0&1&-1\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}1\cdot 2 + 2\cdot 1 + 1\cdot 3\\ 0\cdot 2 + 1\cdot 1 + (-1)\cdot 3\end{pmatrix} = \begin{pmatrix}7\\-2\end{pmatrix}$$

Let us also define the alternative method formally:

Def. (Ax by matrix multiplication rule)

Let A be an $m \times n$-matrix and $x \in \mathbb{R}^n$. The product Ax is defined by:

$$Ax = \begin{pmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{pmatrix} \in \mathbb{R}^m$$

Lemma Let A be an $m \times n$-matrix and $x \in \mathbb{R}^n$.
The two definitions for the product Ax coincide.

Proof Recall the notation:

$$A = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ \vdots \\ a_{m1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ \vdots \\ a_{m2} \end{matrix}} & \dots & \boxed{\begin{matrix} a_{1n} \\ \vdots \\ a_{mn} \end{matrix}} \end{pmatrix} \\ \quad A_1 \quad\;\; A_2 \qquad\quad A_n \end{array}$$

Ax by first definition $= x_1 A_1 + \dots + x_n A_n$

$$= x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Def. of scaler mult.

$$= \begin{pmatrix} x_1 a_{11} \\ \vdots \\ x_1 a_{m1} \end{pmatrix} + \dots + \begin{pmatrix} x_n a_{1n} \\ \vdots \\ x_n a_{mn} \end{pmatrix}$$

Def. of vector addition

$$= \begin{pmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{pmatrix} = Ax \text{ by first definition}$$

$\square$

Def. Two systems of equations are equivalent if they have the same solution space.

Thm. 1.3 Let A be an $m \times n$-matrix and $b \in \mathbb{R}^n$.

Then the equation $Ax = b$ is equivalent to the equation $x_1 A_1 + \dots + x_n A_n = b$, which in turn is equivalent to the linear system with augmented matrix $(A|b)$.

Moreover, either of the above equations/system having a solution is equivalent to $b \in \operatorname{Span}\{A_1, \dots, A_n\}$.

Proof
- The first equivalence is part of the def. of Ax (and the Lemma).
- The second equivalence follows from $Ax = b$ where Ax is written by matrix mult. rule.
- Lastly, let $b \in \operatorname{Span}\{A_1, \dots, A_n\}$.

$\overset{\text{by def.}}{\Longleftrightarrow} \exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$ so that $b = \lambda_1 A_1 + \dots + \lambda_n A_n$

$\Longleftrightarrow$ the equation $b = \underset{\text{unknowns}}{x_1} A_1 + \dots + x_n A_n$ has a solution. $\square$

$(\lambda_1, \dots, \lambda_n)$

From Thm. 1.3 we in particular learn that:

$$Ax = b \text{ has a solution } \Longleftrightarrow b \in \operatorname{Span}\{A_1, \dots, A_n\}$$

Q: Can we solve $Ax = b$ for all $b \in \mathbb{R}^m$?

Ex $n=2, m=3 \Rightarrow A = (A_1, A_2)$, $b \in \mathbb{R}^3$

Is it true that every $b \in \mathbb{R}^3$ is in $\mathrm{Span}\{A_1, A_2\}$?

We know: the span of two vectors in $\mathbb{R}^3$ is either a line or a plane. In either case, there are points $b \in \mathbb{R}^3$ that do not lie in that line or plane.

Hence, it is not true that every $b \in \mathbb{R}^3$ is in $\mathrm{Span}\{A_1, A_2\}$.

The space of $m \times n$ matrices

Thm. 1.4 Let $A \in \mathbb{R}^{m \times n}$. The following statements are equivalent:

(i) for each $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution.

(ii) $\mathrm{Span}\{A_1, \ldots, A_n\} = \mathbb{R}^m$ (every b is in $\mathrm{Span}\{A_1, \ldots, A_n\}$)

(iii) every echelon form of A has a leading entry in every row. (i.e. echelon form has no zero row)

(iv) some echelon form of A has a leading entry in every row.

meaning "at least one"

If a theorem says "the following statements are equivalent" it does not mean every statement is true in every scenario. It means: for every given scenario, either all statements hold, or, none of them holds.

<u>Proof</u> We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii) Assume that for each $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution. Then by Thm 1.3. it follows that:

for each $b \in \mathbb{R}^m$, $b \in \operatorname{Span}\{A_1, \dots, A_n\}$.

But this means that $\operatorname{Span}\{A_1, \dots, A_n\} = \mathbb{R}^m$

(ii) $\Rightarrow$ (iii) Proof by contrapositive: "not (iii) $\Rightarrow$ not (ii)"

Let $\tilde{A}$ be an echelon form of A and assume that $\tilde{A}$ has a row without leading element.

Hence, $\tilde{A}$ has a zero row at the bottom: $\tilde{A} = \begin{pmatrix} & * & & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ 0 & 0 & \dots & 0 \end{pmatrix}$

Choose $\tilde{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n \Rightarrow (\tilde{A} | \tilde{b}) = \left(\begin{array}{cccc|c} & * & & \begin{matrix} * \\ \vdots \\ * \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 & 0 & \dots & 0 & 1 \end{array}\right)$ has no solution (by Thm. 1.2.)

Now: Do the the sequence of matrix operations that transformed A into $\tilde{A}$ backwards to $(\tilde{A} | \tilde{b})$ and we obtain $(A | b)$ where b is some vector in $\mathbb{R}^n$.

Since $(\tilde{A}, \tilde{b})$ has no solution, also $(A | b)$ has no solution.

So by Thm. 1.3: $b \notin \operatorname{Span}\{A_1 \dots A_n\}$.

(iii) $\Rightarrow$ (iv) trivial

(iv) $\Rightarrow$ (i) Assume A has an echelon form for which every row has a leading entry. Let $\tilde{A}$ be this echelon form.

$\Rightarrow$ $\tilde{A}$ has no zero row.

Let $b \in \mathbb{R}^m$ arbitrary and consider the equation $Ax = b$

Use row reduction to bring $(A|b)$ into echelon form: $(\tilde{A}|\tilde{b})$.

Since $\tilde{A}$ has no zero row, $(\tilde{A}|\tilde{b})$ has no row: $(0, \dots, 0 | *)$ with $* \neq 0$

$\Rightarrow$ By Thm. 1.2 : the system with augmented matrix $(A|b)$ has a solution.

$\Rightarrow$ By Thm 1.3 : $Ax = b$ has a solution $\square$

Ex.

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Q.1 : Does the equation $Ax = b$ have a solution for every $b \in \mathbb{R}^3$?

Answer :

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \overset{\text{I-III}}{\sim} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \overset{\text{II-I}}{\sim} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix} \overset{\text{swap}}{\sim} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

So by Thm. 1.4 : there is a solution x for every $b \in \mathbb{R}^3$ $\Leftarrow$ is in echelon form and every row has a leading entry.

Q.2 : Is the solution for each b unique ?

Answer: The 4th column does not contain a leading entry.

Hence, there is a free variable. So, by Thm. 1.2, for every given $b \in \mathbb{R}^3$, the solution is not unique.

Q.3 : Is $\begin{pmatrix} \pi \\ 100 \\ 13.3 \end{pmatrix} \in \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$?

Answer : $\begin{pmatrix} \pi \\ 100 \\ 13.3 \end{pmatrix}$ being in the Span means that $Ax = \begin{pmatrix} \pi \\ 100 \\ 13.3 \end{pmatrix}$ has a solution (by Thm. 1.4). Hence from our answer to Q.1 we know that : yes, the vector does lie in the span.

Observations :

(1.) If # columns of A > # rows of A, then there must always be columns with no leading entries (and hence free variables)

$\Rightarrow$ if there exists a solution, then it is not unique

Or in other words : if $b \in \text{Span}\{A_1 \dots A_n\}$ then there are several ways to build the vector b as a linear combination of the vectors $A_1 \dots A_n$

(2.) If # rows of A > # columns of A, then the echelon form has at most as many leading entries as it has columns.

By Thm 1.4. $\Rightarrow$ cannot solve $Ax=b$ for all $b \in \mathbb{R}^m$.

i.e. there are some b for which the equation $Ax=b$ has no solution x.

Or in other words : there exist vectors $b \in \mathbb{R}^n$ that are not in Span$\{A_1, \ldots A_n\}$.

<u>Ex.</u>

- $A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \\ 6 & 6 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (# columns < # rows)

$$(A \mid b) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 3 & 0 \\ 6 & 6 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{Infinitely many solutions}$$

free variable (and no $(0 \ldots 0 \mid \neq 0)$ row)

- $A = \begin{pmatrix} 1 & 3 & 6 \\ 1 & 3 & 6 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (# rows < # columns)

Solution 1 :

$\Rightarrow$ Span$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \end{pmatrix} \right\}$ = Span$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ a line !

$b \neq \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow b \notin$ Span$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \Rightarrow$ no solution.

Solution 2 :

$$\left(\begin{array}{ccc|c} 1 & 3 & 6 & 1 \\ 1 & 3 & 6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 6 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right) \Rightarrow \text{no solution}$$

Arithmetic of matrix-vector multiplication

Prop. 1.5 Let $A \in \mathbb{R}^{m \times n}$, $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then:

(i) $A(x+y) = Ax + Ay$

(ii) $A(\lambda x) = \lambda(Ax)$

Proof (i)

vector addition

$$A(x+y) = A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) = A\begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

Def. of "Ax"

$$= (x_1+y_1)A_1 + \dots + (x_n+y_n)A_n$$

(Compare to exercise 1.7.)

$$= x_1A_1 + y_1A_1 + \dots + x_nA_n + y_nA_n$$

Rules for vector arithmetic

Def. of "Ax"

$$= (x_1A_1 + \dots + x_nA_n) + (y_1A_1 + \dots + y_nA_n) = Ax + Ay$$

reorder

(ii) analogous $\square$

Def. We denote by I_n the $n \times n$-identity matrix, that is,

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \qquad e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{i-th entry}$$

Moreover, we denote its column vectors by $e_1, \dots, e_n \in \mathbb{R}^n$.

Ex. $I_1 = (1)$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, ...

Important observation:

For every $x \in \mathbb{R}^n$:

$$I_n \cdot x = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = x$$

and in particular $\overset{b}{\overset{\shortparallel}{x}} \in \mathrm{Span}\{\overset{A_1}{\overset{\shortparallel}{e_1}}, \dots, \overset{A}{\overset{\shortparallel}{e_n}}\}$ and hence

$\mathrm{Span}\{e_1, \dots, e_n) = \mathbb{R}^n$.

1.5. The structure of the solution space

Goals:
- describe the solution space of a linear system more geometrically
- learn how some solution spaces with infinitely many elements are objectively larger than other solution spaces with infinitely many elements

<u>Def.</u> $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. We call:

(i) $Ax = 0$ a homogeneous equation

(ii) $Ax = b$ $(b \neq 0)$ an inhomogenous equation

We sometimes also call $Ax = 0$ the homogeneous equation of $Ax = b$.

<u>Thm 1.6</u> Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Assume that $s_b \in \mathbb{R}^n$ is a solution for $Ax = b$.

Then the solution space S of $Ax = b$ equals:

$$S = \{ s = s_b + s_h : s_h \text{ is a solution of } Ax = 0 \}$$

This means: if we know one specific solution s_b of an equation $Ax = b$, and we know we know all solutions s_h of the homogeneous equation $Ax = 0$, then we automatically get all solutions s of $Ax = b$. (And vice versa: $s_h = s - s_b$)

For the proof: remember that "=" means "$\supseteq$" and "$\subseteq$"!

<u>Proof</u> Let s_b be a solution of $Ax = b$.

(See also Exercise 1.7)

$\boxed{\supseteq}$ Let s_h be a solution of $Ax = 0$. And set $s := s_b + s_h$

Then $As = A(s_b + s_h) = As_b + As_h = b + 0 = b \Rightarrow s \in S$

$\boxed{\subseteq}$ Let $s \in S$, that is, $s \in \mathbb{R}^2$ with $As = b$.

Define $\tilde{s} := s - s_b$. Then $A\tilde{s} = As - As_b = b - b = 0$

$$\left.\begin{array}{l} \Rightarrow \tilde{s} \text{ is a solution of } Ax = 0. \\ \text{Moreover, } \quad s = s_b + \tilde{s} \end{array}\right\} \Rightarrow s \in \{ s = s_b + s_h : As_h = 0 \}$$

$\square$

<u>Ex.</u> $A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \quad b = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$

Q: Solution space for $Ax = b$ in param. form?

$$(A|b) = \left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \sim \overset{\text{row reduc.}}{\cdots\cdots} \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x_2 = 2 \\ x_1 = -1 + \frac{4}{3} x_3 \\ x_3 \in \mathbb{R} \text{ free variable} \end{cases} \Rightarrow S = \left\{ \begin{pmatrix} -1 + t\frac{4}{3} \\ 2 \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

We can write: $\begin{pmatrix} -1+t\frac{4}{3} \\ 2 \\ t \end{pmatrix} \overset{*}{=} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$

* split terms that depend on t from those who don't.

Solution space written in parametric form:

$$S = \left\{ \underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{=s_b} + \underbrace{t\begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}}_{\in \text{ solution space of } Ax=0} : t \in \mathbb{R} \right\}$$

We observe: $A\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = b \Rightarrow \underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{=: s_b}$ is a solution of $Ax=b$

$$A\begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A\left(t\cdot\begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ for every t:

$t\cdot\begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}$ is a solution of $Ax=0$

So, in terms of the notation of Thm 1.6., we can choose $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ as s_b and each $t\cdot\begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}$ is an s_h.

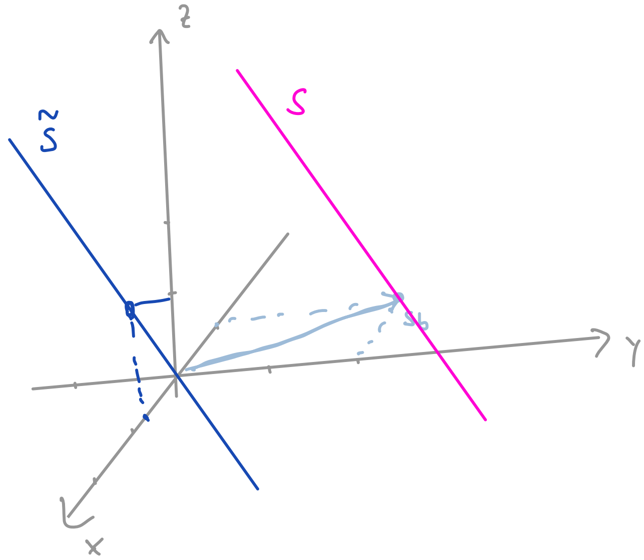
In particular the solution space $\tilde{S}$ for the hom. eq. is

$$\tilde{S} = \left\{ t\begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Geometric interpretation:

$\tilde{S}$ is a line (through the origin) in direction $\begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}$

S is $\tilde{S}$ shifted by $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = S_b$.



Ex. Linear system $(n=1,\ m=3)$

$$\left\{ 2x_1 + 4x_2 - x_3 = 8 \right.$$

$$\Rightarrow (A|b) = \left(2 \quad 4 \quad -1 \mid 8 \right)$$

$$\sim \left(1 \quad 2 \quad -1/2 \mid 4 \right) \qquad \text{REF}$$

$\Rightarrow$ x_2, x_3 are free variables $\rightsquigarrow t, s \in \mathbb{R}$ (or t_1, t_2)

$x_1 = 4 - 2t + \frac{1}{2}s$

$$\Rightarrow S = \left\{ \begin{pmatrix} 4 - 2t + \frac{1}{2}s \\ t \\ s \end{pmatrix} : t, s \in \mathbb{R} \right\}$$

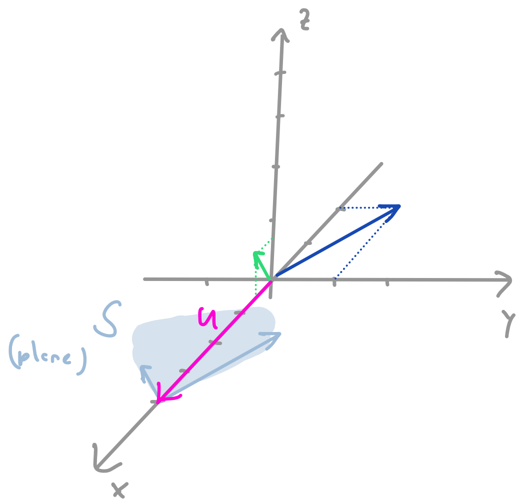
parametric form

$$= \left\{ \underbrace{\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}}_{S_b} + \underbrace{t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}}_{S_h} : t, s \in \mathbb{R} \right\}$$

S is the plane spanned by the vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$ and shifted by $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$.

Therefore, we sometimes also write:

$$S = \underbrace{\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}}_{=u} + \operatorname{Span}\left\{ \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{=v}, \underbrace{\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}}_{=w} \right\}$$



General method for parametric form:

1.) Find REF of $(A|b)$

2.) Express basic variables in terms of free variables ($t_1, \dots, t_k$)

3.) Write $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and plug in all free variables and equations for basic variables in terms of free variables

4.) Decompose x into $\underbrace{s_b}_{\text{part with no free var.}}$ and $\underbrace{\text{a linear combination for } s_h}_{\text{every free variable gives one additive term.}}$.

From the latest example, we see: in order to get a better understanding of solution spaces, we need to learn more about spans of vector sets.

1.6. Linear independence (Chapter 1.7. in the book)

Observation:

- If $v_1, v_2 \in \mathbb{R}^n$ and $v_2 = \lambda v_1$ for some λ in $\mathbb{R}$

  then $\mathrm{Span}\{v_1, v_2\} = \mathrm{Span}\{v_1\} = \mathrm{Span}\{v_2\}$

  if $\lambda \neq 0$

- If $v_1, v_2, v_3 \in \mathbb{R}^n$, $v_3 = a v_1 + b v_2$, $a, b \in \mathbb{R}$

  then $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, v_2\}$

Let us formalize this observation:

set of vectors

<u>Def.</u> Let $v_1, \dots, v_k \in \mathbb{R}^n$ be vectors. We say that $\{v_1, \dots, v_k\}$ are *linearly independent* if:

$$\begin{cases} \text{whenever } \lambda_1 v_1 + \dots + \lambda_k v_k = 0 \text{ for } \lambda_1, \dots, \lambda_k \in \mathbb{R} \\ \text{then it follows that } \lambda_1 = \dots = \lambda_k = 0 \end{cases}$$

equivalently (i.e. in other words):

$$\begin{cases} \text{the } \underline{\text{only}} \text{ solution of the equation } \lambda_1 v_1 + \dots + \lambda_k v_k = 0 \\ \text{is } (\lambda_1, \dots, \lambda_k) = (0, \dots, 0). \end{cases}$$

Otherwise they are called *linearly dependent*.

Ex. $\cdot$ $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -100 \\ 0 \end{pmatrix} \right\}$ are linearly dependent

Because $\underbrace{1}_{=\lambda_1 \neq 0} \cdot \begin{pmatrix} -100 \\ 0 \end{pmatrix} + \underbrace{100}_{=\lambda_2 \neq 0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

- $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent

Consider $\lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$. Solutions?

$(A|b) = \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right)$

$\Rightarrow$ no free variables $\Rightarrow$ only one solution

We already know $(\lambda_1, \lambda_2) = (0,0)$ is a solution

$\Rightarrow$ zero is the only solution $\Rightarrow$ linearly indep.

- the set of vectors that consists of the zero vector:

$\left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$ is linearly dependent

We observe: $\lambda \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ f all $\lambda \in \mathbb{R}$

So e.g. $\underbrace{1}_{\neq 0} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow$ lin. dep.

- the set of vectors that consists of only one vector $v \in \mathbb{R}^n$, $v \neq 0$:

  $\{v\}$ is linearly independent

  $$\lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \lambda v_1 = 0 \\ \vdots \\ \lambda v_n = 0 \end{cases}$$

  $v \neq 0 \Rightarrow$ one of the $v_j \neq 0$

  $\Rightarrow \lambda = 0$ only solution.

  $\Rightarrow$ lin. indep.

<u>General recipe</u> to check a set of vectors $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ for lin. independence:

- Consider $\lambda_1 v_1 + \dots + \lambda_k v_k = 0$, $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ unknowns

- Matrix form: $\underbrace{(v_1 \dots v_k)}_{(n \times k)\text{-matrix}} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

- $\{v_1, \dots, v_k\}$ are lin. indep. $\Leftrightarrow$ system only has one solution (which is zero)

  if and only if

next week!

Some Theorems that provide practical tools to check for linear independence ...

Thm. 1.7. Let $k \geq 2$ and $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$. This set of vectors is linearly dep. if and only if there exists $j \in \{1, \dots, k\}$ such that v_j is a linear combination of the other vectors.

Proof $\boxed{\Rightarrow}$ Assume $\{v_1, \dots, v_k\}$ are linearly dep.

We want to show: one of the v_j can be written in terms of all other v_i s.

linear dependence $\Rightarrow$ the equation $\lambda_1 v_1 + \dots + \lambda_k v_k = 0$ has a non-zero solution

$\Rightarrow \lambda_1, \dots, \lambda_k \in \mathbb{R}$ not all zero, and $\lambda_1 v_1 + \dots + \lambda_k v_k = 0$

Let $j \in \{1, \dots, k\}$ so that $\lambda_j \neq 0$.

resolve for v_j (ok cause $\lambda_j \neq 0$)

$$\Rightarrow v_j = -\frac{1}{\lambda_j}\left(\lambda_1 v_1 + \dots \lambda_{j-1} v_{j-1} + \lambda_{j+1} v_{j+1} + \dots + \lambda_k v_k\right)$$

$$= \underbrace{-\frac{\lambda_1}{\lambda_j} v_1 - \dots - \frac{\lambda_{j-1}}{\lambda_j} v_{j-1} - \frac{\lambda_{j+1}}{\lambda_j} v_{j+1} \dots - \frac{\lambda_k}{\lambda_j} v_k}_{\text{this is a linear comb. of } v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k \ \checkmark}$$

$\boxed{\Leftarrow}$ Similar (slightly easier) → Exercises $\square$

<u>Thm. 1.8</u> • If $k > n$, then any set $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is linearly dependent.

• If $v_j = 0$ for some index $j \in \{1, \dots, k\}$ then $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is lin. dep.

<u>Proof</u>: → Exercises

<u>Thm. 1.9</u> Let $v_1, \dots, v_{k+1} \in \mathbb{R}^n$ vectors

- If $\{v_1, \dots, v_k\}$ are lin. dep., then also $\{v_1, \dots, v_k, v_{k+1}\}$ are dependent.
- If $\{v_1, \dots, v_{k+1}\}$ are lin indep., then also $\{v_1, \dots, v_k\}$ are lin indep.

<u>Proof</u>: First part: Assume $\{v_1, \dots, v_k\}$ are lin. dep.

$\Rightarrow$ $\lambda_1 v_1 + \dots + \lambda_k v_k = 0$ has a non-zero solution: $\lambda_1, \dots, \lambda_n$ and $\lambda_j \neq 0$ for some j

Now consider the equation:

We want to find a non-zero solution

$$(*) \quad \mu_1 v_1 + \dots + \mu_k v_k + \mu_{k+1} v_{k+1} = 0 .$$

Choose $\mu_1 := \lambda_1, \dots, \mu_k := \lambda_k$ and $\mu_{k+1} := 0$

Then, indeed:

- the μ_i are not all zero: $\mu_j = \lambda_j \neq 0$
- $\mu_1 v_1 + \dots + \mu_k v_k + \overbrace{\mu_{k+1}}^{=0} v_{k+1}$
  $= \lambda_1 v_1 + \dots + \lambda_k v_k$
  $= 0$

$\Rightarrow$ Found a non-zero solution for $(*)$

$\Rightarrow$ $\{v_1, \dots v_k, v_{k+1}\}$ are lin. dep.

Second part: $\rightarrow$ Exercises