Linear algebra

Tuesday : class 2h

Thursday : class 2h

Exercise session : 2h

Weekly schedule :

Monday : Homework is uploaded to Moodle with Mini solution

Mon - Thu : you work on your homework

Friday : Full Solutions are uploaded (check your work!)

Special weeks :

Week 4 : Graded homework 1

Week 8 : Graded homework 2

Week 10 : Mock Exam

Week 12 : Graded homework 3

For feedback purposes! They do <u>not</u> count towards your grade in this course.

Tuesday

1.1. Systems of linear equations

<u>Ex.</u> We want to find numbers that solve both of the following linear equations simultaneously:

$$\begin{cases} x + 2y = 1 & (1) \\ -x + y = 0 & (2) \end{cases} \qquad x, y \in \mathbb{R} \text{ unknowns}$$

$$(2) \Rightarrow x = y \underset{\substack{\text{Plug in} \\ \text{eq. (1)}}}{\Rightarrow} 3y = 1 \Rightarrow y = \frac{1}{3} \underset{x=y}{\Rightarrow} x = \frac{1}{3}$$

Verification: (1) $\frac{1}{3} + 2 \cdot \frac{1}{3} = 1$ ✓ (2) $-\frac{1}{3} + \frac{1}{3} = 0$ ✓

In conclusion, the system has one solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$.

Or, in other words: the solution space is $S = \left\{ \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} \right\}$

The general case: a system of m equations and n unknowns

$$(*) \begin{cases} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 & (1) \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{mn} x_n = b_2 & (2) \\ & \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m & (m) \end{cases}$$

We call (*) a system of linear equations and we refer to its different compounds as :

- Unknowns : $x_1, x_2, \dots, x_n \in \mathbb{R}$

- Coefficients (known) : $\left.\begin{matrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ \vdots & & & \\ a_{m1}, & a_{m2}, & \dots, & a_{mn} \end{matrix}\right\} \in \mathbb{R}$

right-hand side

- RHS (known) : $b_1, b_2, \dots, b_m \in \mathbb{R}$

Notation: a_{ij} $\quad 1 \leq i \leq m, \quad 1 \leq j \leq n$

equation unknown

Questions:
- Given the a_{ij} and the b_i, does there exist a solution $(x_1, x_2, \dots, x_n)$?
- If so, is the solution unique ?
- How can we compute the solution(s) ?

Formally, a solution for (*) is a list of real numbers $(s_1, s_2, \dots, s_n)$ so that setting $(x_1, x_2, \dots, x_n) = (s_1, s_2, \dots, s_n)$ makes all equations in (*) true.

General approach: replace $(*)$ by a simpler system of equations that has the same set of solutions.

Ex. We replaced $(*)\begin{cases} x+2y = 1 \\ -x+y = 0 \end{cases}$ by $\begin{cases} x = y \\ 3y = 1 \end{cases}$

by rearranging terms in each equation and by adding one eq. to the other. This did not affect the set of solutions and yielded a simpler to solve system.

To do this for very large systems or to implement it on a computer, it is most handy to use matrix notation for systems of linear equations.

Def. A (real) matrix is a rectangular array that contains numbers in $\mathbb{R}$. We call a matrix an $m \times n$-matrix if it has m rows and n colums.

Ex. $\begin{pmatrix} 1 & 2 & 0 \\ 0 & \pi & -10 \end{pmatrix}$ is a 2×3-matrix.

To each system of linear equations, we can associate a matrix:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{mn}x_n = b_2 \\ \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad (A|b) := \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

the associated matrix — the augmented matrix.

<u>Ex.</u> (*) $\begin{cases} x - 3y + 2z = 1 \\ 4x + y - 5z = 2 \end{cases}$ $(n=3, m=2)$

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 1 & -5 \end{pmatrix} \qquad (A|b) = \left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 4 & 1 & -5 & 2 \end{array}\right)$$

Solving a linear system

Three elementary operations suffice to solve a linear system, and each of them has an analog on the level of the augmented matrix.

System

(OP.1) Swap two equations

(OP.2) Multiply an equation by a non-zero factor ($\lambda \neq 0$)

Ex. $a_{i1}x_1 + \ldots + a_{in}x_n = b_i \quad | \cdot \lambda$

$\Leftrightarrow \lambda a_{i1}x_1 + \ldots + \lambda a_{in}x_n = \lambda b_i$

(OP.3) Add a multiple of some equation to another equation

Ex. Add λ-times the i-th row to the k-th row.

$$\begin{cases} k\text{-th row: } a_{k1}x_1 + \ldots + a_{kn}x_n = b_k \\ i\text{-th row: } a_{i1}x_1 + \ldots + a_{in}x_n = b_i \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} i\text{-th row: same as above} \\ \text{new } k\text{-th row:} \\ (a_{k1}+\lambda a_{i1})x_1 + \ldots + (a_{kn}+\lambda a_{in})x_n = b_k + \lambda b_i \end{cases}$$

Augmented matrix

(OP.1) Swap two rows

(OP.2) Multiply all elements of a row by a non-zero factor ($\lambda \neq 0$)

Ex.
$$\left(\begin{array}{ccc|c} \vdots & & & \vdots \\ a_{i1} & \cdots & a_{in} & b \\ \vdots & & & \vdots \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} \vdots & & & \vdots \\ \lambda a_{i1} & \cdots & \lambda a_{in} & \lambda b \\ \vdots & & & \vdots \end{array}\right)$$

(OP.3) Add a multiple of some row to another row (entry-wise)

Ex.
$$\left(\begin{array}{ccc|c} \vdots & & & \vdots \\ a_{i1} & \cdots & a_{in} & b_i \\ \vdots & & & \vdots \\ a_{k1} & \cdots & a_{kn} & b_k \\ \vdots & & & \vdots \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} \vdots & & & \vdots \\ a_{i1} & \cdots & a_{in} & b_i \\ \vdots & & & \vdots \\ a_{k1}+\lambda a_{i1} & \cdots & a_{kn}+\lambda a_{in} & b_k+\lambda b_i \\ \vdots & & & \vdots \end{array}\right)$$

Example

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 & (2) \\ 5x_1 \phantom{{}-2x_2} - 5x_3 = 10 & (3) \end{cases} \qquad (A|b) = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right) \text{III} - 5\text{I}$$

OP3 $\Updownarrow$ new (3) = (3) − 5·(1)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 & (2) \\ \phantom{x_1 -{}} 10x_2 - 10x_3 = 10 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right)$$

OP3 $\Updownarrow$ new (3) = (3) − 5(2)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 & (2) \\ \phantom{x_1 - 2x_2 +{}} 30x_3 = -30 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{array}\right)$$

OP2 ($\lambda = \frac{1}{30}$) $\Updownarrow$ new (3) = $\frac{1}{30}$ · (3)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ \phantom{x_1 -{}} 2x_2 - 8x_3 = 8 & (2) \\ \phantom{x_1 - 2x_2 +{}} x_3 = -1 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

echelon form!
→ see definition later

$\Updownarrow$ new (2) = (2) + 8·(3)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ 2x_2 = 0 & (2) \\ x_3 = -1 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\Updownarrow \ \text{new}(1) = (1) + 1 \cdot (2) \qquad \downarrow$$

$$\begin{cases} x_1 + x_3 = 0 & (1) \\ 2x_2 = 0 & (2) \\ x_3 = -1 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\Updownarrow \ \text{new}(1) = (1) - 1 \cdot (1) \qquad \downarrow$$

$$\begin{cases} x_1 = 1 & (1) \\ 2x_2 = 0 & (2) \\ x_3 = -1 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\Updownarrow \ \text{new}(2) = \tfrac{1}{2}(2) \qquad \downarrow$$

$$\begin{cases} x_1 = 1 & (1) \\ x_2 = 0 & (2) \\ x_3 = -1 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

reduced echelon form!
→ see definition later

Solution: $(x_1, x_2, x_3) = (1, 0, -1)$

Observation: • Matrix calculation is quicker and less writing effort.

• We can read the solution easily once the matrix has some "stair-like" form and its even easier once we reduce as many entries as possible to zero.

echelon form

1.2. Row reduction and echelon form

Row reduction is algorithmic way of applying the elementary operations to $(A|b)$ to solve the system.

Def. An $m \times n$ - matrix is in echelon form if :

(i) all rows containing only zeros (if there are any) are at the bottom.

(ii) if a row is non-zero, then its left-most non-zero entry (called its leading entry) is in a column to the right of the leading entry of the above row.

The matrix is in reduced echelon form if in addition it also satisfies :

(iii) each leading entry is equal to 1.

(iv) each leading entry is the only non-zero entry in its column.

Ex.

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon ✓

reduced echelon ✓

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon ✓

reduced echelon ✓

$$\begin{pmatrix} \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

↯(ii)

echelon ✗

reduced echelon ✗

$$\begin{pmatrix} 1 & \boxed{1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

↯(iv)

echelon ✓

reduced echelon ✗

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{-3} & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

↯(iii)

echelon ✓

reduced echelon ✗

◯ = leading entry

We observe that it is easy to solve a system if the augmented matrix is in (reduced) echelon form:

Ex. Find the solutions of a system (*) whose augmented matrix $(A|b)$ has the following reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x_1 = 1 \\ x_2 + x_4 = -1 \\ x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = t \in \mathbb{R} \quad \text{free variable} \\ x_1 = 1 \\ x_2 = -1 - t \\ x_3 = -2t \end{cases}$$

$\Rightarrow$ All solutions of $(*)$ are of the form: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1-t \\ -2t \\ t \end{pmatrix}$ for some $t \in \mathbb{R}$.

$$\Rightarrow S = \left\{ \begin{pmatrix} 1 \\ -1-t \\ -2t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Q1: Can every matrix be brought into REF?

Q2: Is the REF of a matrix unique?

$\rightarrow$ 2x Yes!

Thm. 1.1 Every $m \times n$-matrix $\overbrace{\text{is row-equivalent}}^{\text{"existence"}}$ to a $\overbrace{\text{unique}}^{\text{"uniqueness"}}$ matrix in reduced row echelon form. This means that the matrix can be transformed into reduced row echelon form by elementary operations of type (OP. 1), (OP. 2), and (OP. 3)

Proof:
- existence: see algorithm below
- uniqueness: full proof is somewhat tricky.

The row-reduction algorithm (Gauss elimination)

Let A be a general $m \times n$-matrix and let A_j be the j-th column, that is:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \begin{array}{cccc} \underbrace{\phantom{a_{m1}}}_{A_1} & \underbrace{\phantom{a_{m2}}}_{A_2} & & \underbrace{\phantom{a_{mn}}}_{A_n} \end{array} \qquad A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}, \qquad A = (A_1, A_2, \dots, A_n)$$

1st step: For $j=1,\dots,n$ (start at $j=1$, then $j=2,\dots$)

- If $A_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$: proceed to $j+1$
- If $A_j \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$: choose an index i with $a_{ij} \neq 0$.

For said index i: multiply the i-th row of the matrix by $\lambda = \frac{1}{a_{ij}}$ (OP. 2) so it has leading coefficient $=1$.

Then swap i-th and the first row of the matrix (OP. 1)

Then for every other index i for which $a_{ij} \neq 0$: add $\lambda = -a_{ij}$ times the first row to the i-th row (OP. 3)

As a result the column A_j has a 1 on top and only zero else

2nd step: Discard column $1,\dots,j$ as well as row 1 from the matrix. This leaves you with a $(m-1)\times(n-j)$-matrix.

Repeat step 1 for this matrix.

Repeat step 2 until there are no column left.

The algorithm terminates after at most n rounds.

Note: In practice, we do not delete but simply disregard the columns $A_1 \dots A_j$ and the first row in each round.

<u>Example</u> (Gauss elimination)

$$\begin{pmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Step 1: Find first column that is not all zeros: A_2 $(j=2)$

Then find the first entry in that column that is not zero: a_{23} $(i=3)$

Then multiply that entire 3rd row by $\frac{1}{a_{23}} = \frac{1}{2}$:

a_{ij} $i=3$

$$\begin{pmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \cdot \frac{1}{2} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$A_{\cdot j}$ $(j=2)$

Next swap the 3rd row with the first (to move it to the top):

$$\overset{\text{swap}}{\rightsquigarrow} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Next use the leading entry 1 of the new top row to eliminate every other entry of the same column.

Here this means: Kill 3 by adding -3 times the first to the forth row:

$$\overset{IV - 3 \cdot I}{\rightsquigarrow} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Step 2: Delete | disregard 1st row and columns A_1 and A_2

(They are already in as-good-as-can-be shape for REF):

↳ blue staircase

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Then start the process all over for the rest of the matrix.

Step 1

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \text{swap} \sim \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \mathrm{IV} + 3 \cdot \mathrm{I}$$

$$\sim \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \mathrm{V} - \mathrm{I} \sim \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathrm{I} - \mathrm{II}$$

$$\sim \begin{pmatrix} 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \text{start all over}$$

Step 1:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 1: every remaining column is zero $\Rightarrow$ we are done.

And we observe that the matrix is indeed in RREF:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: The Gauss algorithm mainly serves the goal of proving the existence part of Thm. 1.1. In practical examples as the one above, a different, freely chosen order of row operations may be quicker.

Free variables and solutions in parametric form

Let $(A|b)$ be an augmented matrix in reduced echelon form

whose last non-zero row is : $\begin{array}{cccccccc} & & & \text{k-th column} & \text{(k+1) st column} & & & \\ (0, & \dots, & 0, & \boxed{1}, & a_{m(k+1)}, & \dots, & a_{mn}, & b_m) \end{array}$ (m-th row)

This means that the last equation of the system is:

$$x_k + a_{m(k+1)} x_{k+1} + \dots + a_{mn} x_n = b_m$$

or equivalently :

$$x_k = b_m - a_{m(k+1)} x_{k+1} - \dots - a_{mn} x_n$$

$\Rightarrow$ $x_{k+1}, \dots, x_n$ are free variables.

and x_k is expressed in terms of them.

Then we go to the row above :

$$\begin{array}{cccccccccccc} & & & \text{j-th column} & & & & \text{k-th column} & & & & \\ (0, & \dots, & 0, & \boxed{1}, & *, & \dots, & *, & 0, & *, & \dots, & b_{m-1}) \end{array}$$

(m-1)st-row

$(j<k)$

This means that the corresponding equation of the system is:

$$x_j = b_{m-1} - * x_{j+1} - \dots - * x_{k-1} - * x_{k+1} - \dots - * x_n$$

new free variables

free variables from step before

$\Rightarrow$ $x_{j+1}, \dots, x_{k-1}$ are additional free variables.

and x_j is expressed in terms of the free variables (old and new).

Important: x_k is not part of this representation and does not become a free variable in this process!
Same for x_j which is "the second x_k".

Then we move on to the next row above and repeat.

So we solve the system from bottom to top.

We observe:

free variables $\hat{=}$ columns with no leading entry

basic variables $\hat{=}$ columns with leading entry

"corresponds to"

This yields a description of the solution space in parametric form: Say we have a total of $k \geq 0$ free variables.

$\rightsquigarrow$ rename them by $t_1, t_2, \dots, t_k$

Ex.

$$(A|b) = \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & \sin(3) & 0 \\ 0 & 0 & 1 & 0 & \pi/2 & 0 \\ 0 & 0 & 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- Last row: only zeros $\Rightarrow$ no condition on the unknowns (4th row) $x_1, \dots, x_5$ $\Rightarrow$ skip
- 3rd row: this is the last non-zero row.
  Leading entry is in 4th column ($\rightsquigarrow x_4$)
  $\Rightarrow$ every column to the right of 4th column makes for a free variable: x_5

$\Rightarrow$ $x_5 = t_1 \in \mathbb{R}$ free variable

$x_4 = 8t_1$

- 2nd row : leading entry is in 3rd column ($\rightsquigarrow x_3$)

  $\Rightarrow$ every column to the right of 3rd column that does not have a leading entry, makes for a free variable: only x_5

  We already knew x_5 is free $\Rightarrow$ no new free variables

  $x_3 = -\frac{\pi}{2} t_1$

- first row : leading entry is in 1st column ($\rightsquigarrow x_1$)

  $\Rightarrow$ every column to the right of 1st column that does not have a leading entry makes for a free variable : x_2 and x_5

  We already knew x_5 is free , x_2 is new :

  $x_2 = t_2 \in \mathbb{R}$ free

  $x_1 = -\sin(3)\, t_1 + 2t_2$

$\Rightarrow$ A system of linear equations whose augmented matrix is row equivalent to $(A|b)$ has the following solution space:

$$S = \left\{ \left(-\sin(3)\, t_1 + 2t_2,\ t_2,\ -\frac{\pi}{2} t_1,\ 8t_1,\ t_1 \right) : t_1, t_2 \in \mathbb{R} \right\}$$

Q: What do we observe about the relationship between the number of leading entries, free variables and solutions ?

Q: Can you think of an augmented matrix $(A|b)$ in REF for which the system has no solution ?

Thursday

Existence and uniqueness of solutions

Thm. 1.2

(i) A linear system has at least one solution if and only if any echelon form of its augmented matrix has no row of the form $(0, \dots, 0 \mid b)$ with $b \neq 0$.

(ii) If a solution exists, it is unique if and only if there are no free variables.

Proof (i) Part 1: Assume the augmented matrix has a row $(0, \dots, 0 \mid b)$ and $b \neq 0$.

$\Rightarrow 0 \cdot x_1 + \dots + 0 \cdot x_n = b \quad \Rightarrow \quad 0 \neq 0$ ↯

$\Rightarrow$ there cannot be any solution for the system.

(i) Part 2: Assume the augmented matrix has no row $(0, \dots, 0 \mid b)$ with $b \neq 0$.

Then the algorithm from the previous page can be applied to find a solution.

(ii) Part 1: Assume that there exists a solution and there is at least one free variable.

$\Rightarrow$ then that free variable yields a parameter $t \in \mathbb{R}$ and hence there are infinitely many solutions in the solution space S.

(ii) Part 2: Assume that there exists a solution and there is no free variable.

$\Rightarrow$ each column has a leading entry

Hence because it is in REF, all other entries are zero

$\Rightarrow$ all equations are of the form: $x_j = b_j$

which yields the unique solution: $(x_1, \dots, x_n) = (b_1, \dots, b_n)$

$$\left(\begin{array}{cccc|c} 1 & & & 0 & b_1 \\ & 1 & & & b_2 \\ & & 1 & & \vdots \\ 0 & & & \ddots \; 1 & \end{array}\right) \Rightarrow \begin{cases} x_1 = b_1 \\ \vdots \\ x_n = b_n \end{cases}$$

$\square$

1.3 Vector equations

Vectors in $\mathbb{R}^n$ $(n \in \mathbb{N})$

Def A vector in $\mathbb{R}^n$ is an ordered n-tuple of real numbers.

Equivalently, we can say, it is a $(n \times 1)$-matrix.

Notation: $x \in \mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$; $0 \in \mathbb{R}^n$ is the zero-vector $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Sometimes, we refer to them as column vectors in a context where we need to distinguish them from row vectors, i.e. $(1 \times n)$-matrices, $(x_1, x_2, \dots, x_n)$.

However, we usually work with column vectors and refer to them simply as "vectors".

Def (Vector addition and scalar multiplication)

(i) For two vectors $x, y \in \mathbb{R}^n$, we define their sum as:

$$x + y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

(ii) For a vector $x \in \mathbb{R}^n$ and a real number $\lambda \in \mathbb{R}$ (called scalar) we define $\lambda \cdot x = \lambda \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$

(iii) We write: $-x := \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$ and $x - y := x + (-y) = \begin{pmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{pmatrix}$

Rules of vector arithmetic:

Let $u, v, w \in \mathbb{R}^n$ vectors and $\lambda, \mu \in \mathbb{R}$ scalars. Then:

i) $u + v = v + u$

ii) $(u + v) + w = u + (v + w)$

iii) $u + 0 = u$

iv) $u + (-u) = 0$

v) $\lambda(u + v) = \lambda u + \lambda v$

vi) $(\lambda + \mu) u = \lambda u + \mu u$

vii) $\lambda(\mu u) = (\lambda \mu) u$

viii) $1 \cdot u = u$

Def. For vectors $u_1, \dots, u_k \in \mathbb{R}^n$ and scalars $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ the vector

$$b = \lambda_1 u_1 + \dots + \lambda_k u_k = \sum_{i=1}^{k} \lambda_i u_i$$

is called a linear combination of $u_1, \dots, u_k$.

The set of all linear combinations of $u_1, \dots, u_k$, is called the span of $u_1, \dots, u_k$:

$$\operatorname{Span}\{u_1, \dots, u_k\} := \left\{ \sum_{i=1}^{k} \lambda_i u_i \;\middle|\; \lambda_1, \dots, \lambda_k \in \mathbb{R} \right\}$$

Ex. $b = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$ is in the span of $u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

Why? → because b can be written as a linear combination of u_1 and u_2:

$$\underbrace{\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}}_{=b} = \overset{\lambda_1}{\overset{\parallel}{1}} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{=u_1} + \overset{\lambda_2}{\overset{\parallel}{2}} \cdot \underbrace{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}_{=u_2}$$

Question: How to solve this kind of problem in general?

I.e., how to determine whether a given vector $b \in \mathbb{R}^n$ is in the span of given vectors $u_1, \dots, u_k$?

$\rightarrow$ We have to solve:

unknowns

$$b = \lambda_1 u_1 + \ldots + \lambda_k u_k \quad \text{for } \lambda_1, \ldots, \lambda_k \in \mathbb{R}$$

$$\Leftrightarrow \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \lambda_1 \begin{pmatrix} u_{11} \\ \vdots \\ u_{n1} \end{pmatrix} + \ldots + \lambda_k \begin{pmatrix} u_{1k} \\ \vdots \\ u_{nk} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 u_{11} \\ \vdots \\ \lambda_1 u_{n1} \end{pmatrix} + \ldots + \begin{pmatrix} \lambda_k u_{1k} \\ \vdots \\ \lambda_k u_{nk} \end{pmatrix}$$

by def. of scalar multiplication

$$\Leftrightarrow \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \lambda_1 u_{11} + \ldots + \lambda_k u_{1k} \\ \vdots \qquad\qquad \vdots \\ \lambda_1 u_{n1} + \ldots + \lambda_k u_{nk} \end{pmatrix}$$

by def. of vector addition

$$\Leftrightarrow \begin{cases} b_1 = u_{11} \lambda_1 + \ldots + u_{1k} \lambda_k \\ \vdots \\ b_n = u_{n1} \lambda_1 + \ldots + u_{nk} \lambda_k \end{cases}$$

this is a linear system of n equations and k unknowns

Augmented matrix:

$$\left(\begin{array}{ccc|c} u_{11} & \cdots & u_{1k} & b_1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \cdots & u_{nk} & b_n \end{array}\right)$$

$n \times (k+1)$ - matrix

$\rightsquigarrow$ solve by elimination