7.3 The matrix $\exp(tA)$ and ODEs

Consider the following ordinary differential equation (ODE):

$$x'(t) = a\, x(t) \qquad (*)$$

Then we know: every multiple of ce^{at} $(c \in \mathbb{R})$ is a solution:

Set $x(t) = c\, e^{at}$ for some constant $c \in \mathbb{R}$

$$\Rightarrow x'(t) = (c\, e^{at})' = a\, c\, e^{at} = a\, x(t) \qquad (**)$$

So we can even find a solution for $(*)$ if we also prescribe an initial value. Let $t_0, c_0 \in \mathbb{R}$.

Say we want that: $x(t_0) = c_0$

then we can just find a suitable constant c:

$$x(t_0) = c\, e^{at_0} = c_0 \Rightarrow c = c_0\, e^{-at_0}$$

So $x(t) = (c_0 e^{-at_0})\, e^{at}$ is a solution for the initial value problem $\begin{cases} a\,x(t) = x'(t) \\ x(t_0) = c_0 \end{cases}$

The reason why $(**)$ works is the definition of exp:

$$e^s := \exp(s) = \sum_{k=0}^{\infty} \frac{s^k}{k!}$$

$$\Rightarrow (e^s)' \underset{\substack{\uparrow \\ \text{Same Thing} \\ \text{from Analysis}}}{=} \sum_{k=0}^{\infty} \left(\frac{s^k}{k!} \right)' = \sum_{k=0}^{\infty} k \frac{s^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{s^{k-1}}{(k-1)!} \underset{\substack{\uparrow \\ j = k-1}}{=} \sum_{j=0}^{\infty} \frac{s^j}{j!} = e^s$$

$$\overset{s = at}{\Rightarrow} \quad (e^{at})' \underset{\substack{\uparrow \\ \text{Chain rule}}}{=} e^{at} (at)' = a e^{at}$$

What if we want to solve a system of n linear ODEs ?

e.g.
$$\begin{cases} x_1'(t) = x_1(t) + 2 x_2(t) \\ x_2'(t) = 3 x_1(t) + 2 x_2(t) \end{cases}$$

or more generally : for constants $a_{ij} \in \mathbb{R}$,

$$(*) \quad \begin{cases} x_1'(t) = a_{11} x_1(t) + \dots + a_{1n} x_n(t) \\ \vdots \\ x_n'(t) = a_{n1} x_1(t) + \dots + a_{nn} x_n(t) \end{cases}$$

Such a system of linear ODEs can be written in matrix-vector-form :

Define $x(t) := \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ and $x'(t) := \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$

and $A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$

Then (*) is equivalent to $\boxed{Ax(t) = x'(t)}$

This is pretty much just like (*) : $a\,x(t) = x(t)$

So we would like to use same approach!

Def. Let $A \in \mathbb{R}^{n\times n}$. Define $\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k \quad \in \mathbb{R}^{n\times n}$

Rmk
- Methods from analysis show that this infinite sum always convergences.
- For the zero matrix $0 \in \mathbb{R}^{n\times n}$: $\exp(0) = I_n$
- Many of the usual computation rules for exp still hold, e.g. $\exp(A)\exp(B) = \exp(A+B)$
  $\exp(A)\exp(-A) = \exp(A-A) = \exp(0) = I_n$

Lemma 7.5 Consider the function $f: \mathbb{R} \to \mathbb{R}^{n\times n}$ defined by $f(t) = \exp(tA)$. This function is differentiable and $f'(t) = A\exp(tA)$.

The computational part of the proof is just like in the case of (**) but it requires more justifications in some steps.
(Proof omitted.)

Thm 7.6 Let $A \in \mathbb{R}^{n \times n}$, $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$.

Then the function $x : \mathbb{R} \to \mathbb{R}^n$ defined by $x(t) = \exp(tA)\,C$

is a solution of $x'(t) = A\,x(t)$

Proof $x(t) := \exp(tA)\,C \underset{\substack{\text{Lemma 7.5} \\ \text{and chain rule}}}{\Rightarrow} x'(t) = A\exp(tA)\,C = A\,x(t)$ $\square$

Cor 7.7 Consider the initial value problem

$$\begin{cases} x'(t) = A\,x(t) \\ x(t_0) = C \end{cases}$$ for constants $t_0 \in \mathbb{R}$, $C \in \mathbb{R}^n$

Then $x(t) = \exp(tA)\,(\exp(-t_0 A)\,C)$ is a solution.

In particular: if $t_0 = 0$, $x(t) = \exp(tA)\,C$ is a solution.

Proof By Thm. 7.6: $x(t) = \exp(tA)\,\underbrace{(\exp(t_0 A)\,C)}_{= \tilde{C} \in \mathbb{R}^n}$

solves $x'(t) = A\,x(t)$.

Moreover: $x(t_0) = \underbrace{\exp(t_0 A)\exp(-t_0 A)}_{= I_n}\,C = C$

$\square$

Since $\exp(tA)$ allows us to easily solve systems of linear ODE, we now want to think about how to compute $\exp(tA)$ for a given matrix $A \in \mathbb{R}^{n\times n}$.

<u>Ex.</u> $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad \exp(tA) = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = ?$

$A^k = \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}$. Hence:

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} \frac{t^k}{k!} 2^k & 0 \\ 0 & \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} 2^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} 3^k \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}$$

<u>Prop. 7.8</u> $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \Rightarrow \exp(tD) = \begin{pmatrix} e^{d_1 t} & & \\ & \ddots & \\ & & e^{d_n t} \end{pmatrix}$

<u>Proof</u> $D^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} \Rightarrow \frac{1}{k!} t^k D^k = \begin{pmatrix} \frac{1}{k!} t^k \lambda_1^k & & \\ & \ddots & \\ & & \frac{1}{k!} t^k \lambda_n^k \end{pmatrix}$

$$\Rightarrow \overbrace{\sum_{k=0}^{\infty} \frac{1}{k!} t^k D^k}^{=\exp(tD)} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} t^k \lambda_1^k & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty} \frac{1}{k!} t^k \lambda_n^k \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}$$

$\square$

<u>Ex</u> $A = \begin{pmatrix} 1 & 4 \\ \pi & -13 \end{pmatrix} \rightsquigarrow A^k = ?!$

<u>Q:</u> Method for computing $\exp(tA)$?

$\rightarrow$ diagonalization.

<u>Prop. 7.9</u> If $A \in \mathbb{R}^{n\times n}$ is diagonalizable, $A = PDP^{-1}$

then: $\exp(tA) = P \exp(tD) P^{-1}$

<u>Proof</u> $$\exp(tA) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \overbrace{(PDP^{-1})^k}^{= PD^kP^{-1}}$$

$$= P\left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k D^k\right) P^{-1} = P \exp(tD) P^{-1}$$ $\square$

This means:
- a diagonalization of A automatically gives a diagonalization of $\exp(tA)$.
- Prop. 7.8 and 7.9 together provide a method for the computation of $\exp(tA)$.

<u>Ex.</u> Find a solution for the system

$$(\circledast)\quad \begin{cases} x_1'(t) = x_1(t) + x_2(t) \\ x_2'(t) = x_1(t) + x_2(t) \\ x_3'(t) = 3x_3(t) \end{cases}$$

with initial conditions $\begin{cases} x_1(0) = 2 \\ x_2(0) = 4 \\ x_3(0) = 0 \end{cases}$

<u>Solution</u>

(1) Writing the system in matrix-vector form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}}_{=A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ x_1(t) + x_2(t) \\ 3x_3(t) \end{pmatrix} \rightsquigarrow A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

So $(\circledast) \iff Ax = x'$

(2) Compute $\exp(tA)$

$$\chi_A(\lambda) = \det(\lambda I_n - A) = \begin{pmatrix} \lambda-1 & -1 & 0 \\ -1 & \lambda-1 & 0 \\ 0 & 0 & \lambda-3 \end{pmatrix}$$

$$= (\lambda-3)\cdot\det\begin{pmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{pmatrix} = (\lambda-3)\left((\lambda-1)^2-1\right)$$

$$= (\lambda-3)(\lambda - 2\lambda + 1 - 1) = (\lambda-3)(\lambda-2)\lambda$$

$\Rightarrow$ eigenvalues $\lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_1 = 0$

eigenvectors :

$$\lambda_1 = 3 \ : \ \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}_{=v_1} = 3 \underbrace{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}_{=v_1} \rightsquigarrow v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_1 = 2 \ : \ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = 2 \cdot \begin{pmatrix} \\ \\ \\ \end{pmatrix} \rightsquigarrow v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_1 = 0 \ : \ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = 0 \cdot \begin{pmatrix} \\ \\ \\ \end{pmatrix} \rightsquigarrow v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

So :

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

$$\Rightarrow \quad A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

$$\exp(tA) \overset{\text{Prop. 7.9}}{=} P\exp(tD)P^{-1} \overset{\text{Prop. 7.8}}{=} P\begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

(3) Compute solution for (&)

By Thm. 7.6. for each constant $C \in \mathbb{R}^3$: $x(t) = \exp(tA)C$ is a solution of $Ax = x'$

By Cor 7.7. if we pick $C = \underbrace{\exp(0 \cdot A)}_{= I_n}\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$

then $x(t) = \exp(tA)C$ also satisfies the initial value condition: $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$.

So we compute: $x(t) = \exp(tA)C$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & 1 \end{pmatrix}\overbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}}^{= \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}}_{= \begin{pmatrix} 0 \\ 3e^{2t} \\ 1 \end{pmatrix}} = \begin{pmatrix} 3e^{2t} - 1 \\ 3e^{2t} + 1 \\ 0 \end{pmatrix}$$

So $\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 3e^{2t} - 1 \\ 3e^{2t} + 1 \\ 0 \end{pmatrix}$ is a solution of (&) for the initial conditions $x_1(0) = 2$, $x_2(0) = 4$, $x_3(0) = 0$.

$\underline{\text{Ex.}}$ A linear ODE of higher order

Find a non-zero solution for the ODE:

$$(\Delta) \quad y''(t) + 3y'(t) - 4y(t) = 0$$

(1) Writing the system in matrix-vector form:

Let $x = \begin{pmatrix} y \\ y' \end{pmatrix} \Rightarrow x' = \begin{pmatrix} y' \\ y'' \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} y' \\ 4y - 3y' \end{pmatrix}$

$(\Delta) \Rightarrow y'' = 4y - 3y'$

We want A so that $Ax = x'$:

$$\underbrace{\begin{pmatrix} \quad & \quad \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} y \\ y' \end{pmatrix}}_{=x} = \underbrace{\begin{pmatrix} y' \\ 4y - 3y' \end{pmatrix}}_{=x'} \leadsto A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$$

(2) Compute $\exp(tA)$

Diagonalize A:

$$A = \underbrace{\begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix}}_{=P} \underbrace{\begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}}_{=D} \underbrace{\begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}}_{=P^{-1}}$$

$$\Rightarrow \exp(tA) = P \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{t} \end{pmatrix} P^{-1}$$

(3) Compute solution for $Ax = x'$

By Thm. 7.6. for each constant $C \in \mathbb{R}^2$: $x(t) = \exp(tA)C$ is a solution of $Ax = x'$

So we choose for example $C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then $\exp(tA)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution of $Ax = x'$

$$x(t) = \exp(tA)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = P\begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{t} \end{pmatrix}P^{-1}$$

$$= \begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{t} \end{pmatrix}\underbrace{\begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= \begin{pmatrix} -\frac{1}{5} \\ \frac{4}{5} \end{pmatrix}}}_{= \begin{pmatrix} -\frac{1}{5}e^{-4t} \\ \frac{4}{5}e^{t} \end{pmatrix}} = \frac{1}{5}\begin{pmatrix} e^{-4t} + 4e^{t} \\ -4e^{-4t} + 4e^{t} \end{pmatrix}$$

(4) Solution for (Δ)

$x(t)$ is a solution for $Ax = x'$ $\overset{x(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}}{\Longleftrightarrow}$ $y(t) = x_1(t)$ is a solution for (Δ)

$\Rightarrow$ $y(t) = \frac{1}{5}(e^{-4t} + 4e^{t})$ is a non-zero solution for (Δ).

(5) Check

$$\left.\begin{aligned} y' &= -\frac{4}{5}e^{-4t} + \frac{4}{5}e^{t} \\ y'' &= \frac{16}{5}e^{-4t} + \frac{4}{5}e^{t} \end{aligned}\right\} \Rightarrow y''(t) + 3y'(t) - 4y(t)$$

$$= \frac{16}{5}e^{-4t} + \frac{4}{5}e^{t} + 3\left(-\frac{4}{5}e^{-4t} + \frac{4}{5}e^{t}\right) - 4\left(\frac{1}{5}(e^{-4t} + 4e^{t})\right)$$

$$= 0 \quad \checkmark$$