Recall: If $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $v \in \mathbb{R}^n$, $v \neq 0$ satisfying $Av = \lambda v$ then we call λ an eigenvalue of A and v an eigenvector of A for λ.

Def. Let $A \in \mathbb{R}^{n\times n}$ a matrix. We define the spectrum of A to be the set of all its eigenvalues:

$$\sigma(A) = \{ \lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A \}$$

Def. Let $A \in \mathbb{R}^{n\times n}$ a matrix and λ an eigenvalue of A.

$\Rightarrow$ λ is a root of the polynomial $\chi_A(\lambda)$ (by Prop. 5.2)

We call the multiplicity of the root the algebraic multiplicity of the eigenvalue λ of A.

Ex. $$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \lambda I_n - A = \begin{pmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda-\pi & 0 \\ 0 & 0 & \lambda+1 \end{pmatrix}$$

$$\Rightarrow \chi_A(\lambda) = (\lambda+1)^2(\lambda-\pi)$$

$$\sigma(A) = \{-1, \pi\}$$

The algebraic multiplicity of -1 is 2

The algebraic multiplicity of π is 1

Def. Let $A \in \mathbb{R}^{n \times n}$ a matrix and $\lambda \in \sigma(A)$.

We define the eigenspace of A for λ to be:

$$E_A(\lambda) := \operatorname{Ker}(\lambda I_n - A)$$

The geometric multiplicity of λ (for A) is: $\dim E_A(\lambda)$.

Rmk

- $E_A(\lambda) \subseteq \mathbb{R}^n$ is a subspace (because it is the kernel of a matrix)
- $E_A(\lambda) = \{\text{eigenvectors of } A \text{ for } \lambda\} \cup \{0\}$.
- geom. mult. = # indep. eigenvectors of A for λ.

Ex. In the previous example:

$$E_A(-1) = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\} \Rightarrow$$ The geometric multiplicity of -1 (for A) is 2

$$E_A(\pi) = \operatorname{Span}\left\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\} \Rightarrow$$ The geometric multiplicity of π (for A) is 1

Ex.

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \pi & 1 \\ 0 & 0 & 0 & \pi \end{pmatrix} \Rightarrow$$

eigenvalues:	$\lambda = -1$	$\lambda = \pi$
alg. mult:	2	2
geom mult:	2	1

Thm 5.4 Let $A \in \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_k$ distinct eigenvalues of A. For each i, let v_i be an eigenvector of the eigenvalue λ_i. Then: $\{v_1, \dots, v_k\}$ is independent.

Proof by *induction*:

$k=1$: λ_1 is an eigenvalue and v_1 is an eigenvector for λ_1

$\Rightarrow v_1 \neq 0 \Rightarrow \{v_1\}$ lin. indep.

$k-1 \to k$: Assume $\{v_1, \dots, v_{k-1}\}$ are indep

To show: $\{v_1, \dots, v_{k-1}, v_k\}$ are indep.

Let $\mu_1 v_1 + \dots + \mu_{k-1} v_{k-1} + \mu_k v_k = 0$.

Multiply both sides by $A - \lambda_k I_n$:

$$(A - \lambda_k I_n)(\mu_1 v_1 + \dots + \mu_{k-1} v_{k-1} + \mu_k v_k) = 0.$$

We compute:

- $(A - \lambda_k I_n)(\mu_k v_k) = \mu_k (A - \lambda_k I_n) v_k$
$= \mu_k (\underbrace{A v_k}_{= \lambda_k v_k} - \underbrace{\lambda_k I_n v_k}_{= \lambda_k v_k}) = 0$

- For $i \neq k$:

$$(A - \lambda_k I_n)(\mu_i v_i) = \mu_i (\underbrace{A v_i}_{= \lambda_i v_i} - \underbrace{\lambda_k I_n v_i}_{= \lambda_k v_i}) = \mu_i (\lambda_i - \lambda_k) v_i$$

Hence: $\mu_1(\lambda_1-\lambda_k)v_1+\dots+\mu_{k-1}(\lambda_{k-1}-\lambda_k)v_{k-1}=0$

By lin. indep of $\{v_1 \dots v_{k-1}\}$ all coefficients are zero:

$$\mu_1\underbrace{(\lambda_1-\lambda_k)}_{\neq 0}=\dots=\mu_{k-1}\underbrace{(\lambda_{k-1}-\lambda_k)}_{\neq 0 \text{ cause } \lambda_1 \dots \lambda_k \text{ distinct}}=0$$

$\Rightarrow \mu_1=\dots=\mu_{k-1}=0 \qquad \Rightarrow \mu_k=0$ ($\llcorner v_k \neq 0$) $\square$

Prop. 5.5 Let $A \in \mathbb{R}^{n\times n}$.

(1) If A is invertible then: $\lambda \in \sigma(A) \Leftrightarrow \frac{1}{\lambda} \in \sigma(A^{-1})$

(2) $\sigma(A)=\sigma(A^T)$

(3) $\lambda \in \sigma(A) \Rightarrow \lambda^k \in \sigma(A^k)$

Proof: exercise! $\rightsquigarrow$ Ex. 10.7

Real vs. complex eigenvalues

In this course, we focus on real eigenvalues $\lambda \in \mathbb{R}$. However, since eigenvalues are roots of polynomial (namely the polynomial $\chi_A(\lambda)$) it would be silly not to at least mention complex eigenvalues.

(see Thm. 4.23: fundamental theorem of algebra)

Assume $A \in \mathbb{R}^{n \times n}$ is a matrix and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ a non-real eigenvalue. Then its eigenvectors v must also have at least one non-real entry:

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be an eigenvector for λ

$$\Rightarrow A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \vdots \\ a_{i1} x_1 + \ldots + a_{in} x_n \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \lambda x_i \\ \vdots \end{pmatrix}$$

If all x_i were real then $a_{i1} x_1 + \ldots + a_{in} x_n \in \mathbb{R}$

but $\lambda x_i \in \mathbb{C} \setminus \mathbb{R}$. $\lightning$

<u>Ex.</u> $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ <u>Q</u>: eigenvalues of A?

$$\chi_A(\lambda) = \det(\lambda I_n - A)$$
$$= \det \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 + 1$$

The roots of $\chi_A(\lambda)$ are $\lambda = i$ and $\lambda = -i$

$\Rightarrow$ A has no real eigenvalues

But it has complex eigenvalues $\lambda = i$ and $\lambda = -i$

Notice that the linear map $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by A is the counter-clockwise rotation by angle $\frac{3\pi}{2}$.

Ex. More generally, the matrix $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ as a linear map, is the counter-clockwise rotation by angle $\alpha \in [0, 2\pi)$

- If $\alpha = 0 \Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda = 1$ is the only eigenvalue and every $v \neq 0$ is an eigenvector for 1
- If $\alpha = \pi \Rightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda = -1$ is the only eigenvalue and every $v \neq 0$ is an eigenvector for -1
- If $\alpha \notin \{0, \pi\}$ then, A has no real eigenvalues and eigenvectors.

(Geometric reason: there cannot be a vector $v \neq 0$ s. that v and Av have the same direction.)

→ see HW 10, exercise 10.10

! For the exam: you must be able to find all eigenvalues of a given matrix $A \in \mathbb{R}^{n \times n}$ (also the complex eigenvalues) but only for the real eigenvalues you must be able to find its eigenvectors.

A first glimpse at eigenbases and diagonalization

Ex. Consider the matrix $A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$

$$\Rightarrow \lambda I_n - A = \begin{pmatrix} \lambda+3 & 0 & 0 \\ 0 & \lambda-2 & -2 \\ 0 & -2 & \lambda-2 \end{pmatrix}$$

$$\begin{aligned} \chi_A(\lambda) = \det(\lambda I_n - A) \; (\lambda+3) \det \begin{pmatrix} \lambda-2 & -2 \\ -2 & \lambda-2 \end{pmatrix} \\ &= (\lambda+3)\left((\lambda-2)^2 - 4\right) \\ &= (\lambda+3)(\lambda^2 - 4\lambda + 4 - 4) \\ &= (\lambda+3)(\lambda^2 - 4\lambda) = (\lambda+3)(\lambda-4)\lambda \end{aligned}$$

$$\Rightarrow \sigma(A) = \{-3, 0, 4\}$$

We have found 3 eigenvalues: $\lambda_1 = -3$, $\lambda_2 = 0$, $\lambda_3 = 4$

Each eigenvalue must have (at least) one eigenvector.

Let v_i be an eigenvector for λ_i.

By Thm. 5.4, we know: $\{v_1, v_2, v_3\}$ are independent.

$\Rightarrow \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

Def. Let $A \in \mathbb{R}^{n \times n}$. If there exist n lin. indep. eigenvectors of A, we call them an eigenbasis of A.

Rmk. By Thm. 5.4, if $A \in \mathbb{R}^{n\times n}$ has n distinct real eigenvalues then A has an eigenbasis.

Ex. (continuation of above example)

Find the eigenbasis of A, i.e. find an eigenvector for each eigenvalue.

In this example, they can easily be guessed. (If you cannot guess them, compute them as usual by solving the respective linear system.)

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{=v} = \underset{\substack{\lambda_1=-3 \\ \lambda_2=0 \\ \lambda_3=4}}{\lambda} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{=v} \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are suitable eigenvectors

$$\Rightarrow \mathcal{B} := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is an eigenbasis of } A.$$

Now, let us express the linear map $f(x) = Ax$ in terms of $\mathcal{B}$. the basis $\mathcal{B}$. I.e. $f: \mathbb{R}^3 \to \mathbb{R}^3$ and we change basis (from standard to $\mathcal{B}$ in the domain and codomain.

Recall from week 8 : $P_B := (b_1 \dots b_n) \Rightarrow x = P_B \cdot [x]_B$

$$P_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow P_B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

So the matrix M of f in terms of $\mathcal{B}$ is :

$M = P_B^{-1} A P_B$ since :

$$\begin{tikzcd} v \ \mathbb{R}^3 \arrow[r, "A\cdot"] & \mathbb{R}^3 \ f(v) = Av \arrow[d, "P_B^{-1}"] \\ {[v]_B} \ \mathbb{R}^3 \arrow[u, "P_B"] \arrow[r, "M\cdot"] & \mathbb{R}^3 \ M\cdot[v]_B = [f(v)]_B \end{tikzcd}$$

Compute M :

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Observe : M is a diagonal matrix and the diagonal elements of M are the eigenvalues of A.

Let us try to understand what happened here,
i.e., why does M happen to be diagonal with the eigenvalues of A on its diagonal?

Let $A \in \mathbb{R}^{n\times n}$ and assume A has an eigenbasis $\mathcal{B} = \{b_1, \dots, b_n\}$.
Let $M = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$. (And let λ_i be the eigenvalue for b_i).

Where does the matrix $M = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$ map the vectors e_i of the standard basis?

Recall:
$P_{\mathcal{B}} [v]_{\mathcal{B}} = v \quad \forall v \in \mathbb{R}^n$
Choose $v = b_i$ then
$P_{\mathcal{B}} \underbrace{[b_i]_{\mathcal{B}}}_{= e_i} = b_i$

$$M e_i = (P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}) e_i = (P_{\mathcal{B}}^{-1} A) \underbrace{(P_{\mathcal{B}} e_i)}_{= b_i}$$

by assumption, b_i is an eigenvector of the eigenvalue λ_i of A: $A b_i = \lambda_i b_i$

$$= (P_{\mathcal{B}}^{-1} A) b_i = P_{\mathcal{B}}^{-1} \underbrace{(A b_i)}_{= \lambda_i b_i}$$

Because $P_{\mathcal{B}} b_i = e_i$ (see above)

$$P_{\mathcal{B}}^{-1} (\lambda_i b_i) = \lambda_i \underbrace{P_{\mathcal{B}}^{-1} (b_i)}_{= e_i}$$

$\Rightarrow \quad M e_i = \lambda_i e_i$

(i.e. λ_i is an eigenvalue for M and e_i an eigenvector.)

Observe: $M e_i$ is the i-th column of M

$\Rightarrow$ the i-th column of M is: $M e_i = \lambda_i e_i = \begin{array}{cc} \begin{pmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow i \\ \\ \\ \end{array} \end{array}$

$$\Rightarrow \quad M = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{pmatrix}$$

5.3. Diagonalization

Def. Let $A, M \in \mathbb{R}^{n\times n}$. We say that A and M are *similar* if $\exists\, P \in \mathbb{R}^{n\times n}$ invertible so that $M = P^{-1} A P$.

Rmk. We can always view this as a change of basis:

If $M = P^{-1} A P$, then M is the matrix that represents A in the basis that consists of the columns of P.

(Since P is invertible, its columns always build a basis of $\mathbb{R}^n$.)

Prop. 5.6 Let $A, M \in \mathbb{R}^{n\times n}$ be similar $(M = P^{-1} A P)$, then:

(1) $\chi_A(\lambda) = \chi_M(\lambda)$

(2) $\sigma(A) = \sigma(M)$

(3) v is an eigenvector of M for λ

$\Leftrightarrow$ Pv is an eigenvector of A for λ

Proof (1) $\chi_M(\lambda) = \det(\lambda I_n - M) = \det(\lambda I_n - P^{-1} A P)$

Observe: $P^{-1}(\lambda I_n) P = \lambda (P^{-1} I_n P) = \lambda P^{-1} P = \lambda I_n$

So: $$\chi_M(\lambda) = \det(P^{-1}(\lambda I_n) P - P^{-1} A P)$$
$$= \det(P^{-1}(\lambda I_n - A) P)$$

$\det(P^{-1}) = \frac{1}{\det(P)}$

$$= \det(P^{-1}) \det(\lambda I_n - A) \det(P)$$

$$= \det(\lambda I_n - A) = \chi_A(\lambda)$$

(2) Follows from (1) immediately.

(3.) Notice that by symmetry of the definition of similarity, it suffices to check one direction. We do $\boxed{\Rightarrow}$:

Assume that v is an eigenvector of M for λ,

i.e, $Mv = \lambda v$.

To show: Pv is an eigenvector of A for λ,

i.e, $A(Pv) = \lambda(Pv)$.

Since $M = P^{-1}AP$, we have $PM = AP$

So $APv = PMv = P(\lambda v) = \lambda \cdot Pv$ $\square$

Q: We learned that:

If A and M are similar, then $\chi_A(\lambda) = \chi_M(\lambda)$.

Does the converse hold?

→ No! Ex $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

! Warning: Similarity of two matrices is not at all the same as row equivalence! Do not confuse the two notions.

Def. A matrix $A \in \mathbb{R}^{n \times n}$ is called diagonalizable if it is similar to a diagonal matrix.

Q: Under which circumstances is a matrix A diagonalizable?

Thm 5.7 A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if it has an eigenbasis.

In this case, let $\mathcal{B} = \{b_1 \dots b_n\}$ be an eigenbasis of A and λ_i the eigenvalue for the eigenvector b_i for every i. Then:

$$D = P^{-1} A P \quad \text{where } P = P_{\mathcal{B}} = (b_1 \dots b_n) \text{ and } D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Proof We already proved $\boxed{\Leftarrow}$ as well as the bottom part ("in this case") of the theorem in the explanation of the above example.

$\boxed{\Rightarrow}$ Assume that A is diagonalizable

$\Rightarrow$ A is similar to a diagonal matrix

$\Rightarrow$ $\exists$ diagonal matrix $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ and an invertible matrix P

s. that $A = PDP^{-1}$ (or $D = P^{-1} A P$)

By Thm. 5.3 $\Rightarrow$ $\sigma(D) = \{d_1, \dots, d_n\}$ with eigenvectors $e_1, \dots, e_n$.

By Prop. 5.6 $\Rightarrow$ $\sigma(A) = \{d_1, \dots, d_n\}$ with eigenvectors $Pe_1, \dots, Pe_n$

Since P is invertible and $\{e_1 \dots e_n\}$ is a basis of $\mathbb{R}^n$

$\Rightarrow$ $\{Pe_1 \dots Pe_n\}$ is a basis of $\mathbb{R}^n$

Since the Pe_i are eigenvectors of A

$\Rightarrow$ $\{Pe_1 \dots Pe_n\}$ is an eigenbasis for A. $\square$

<u>Cor. 5.8</u> Let $A \in \mathbb{R}^{n \times n}$ diagonalizable then the diagonal matrix D that is similar to A is unique up to the order of elements on the diagonal.

<u>Cor. 5.9</u> If $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues, then A is diagonalizable.

(Proofs are easy and omitted.)

<u>Q</u>: If $A \in \mathbb{R}^{n \times n}$ does not have n distinct eigenvalues, can it still be diagonalizable?

$\longrightarrow$ sometimes yes, sometimes no.

e.g. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

next example

Ex. $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$

1.) Compute $\chi_A(\lambda)$ and find roots:

$$\chi_A(\lambda) = \det(\lambda I_3 - A) = \dots = \lambda^3 + 3\lambda^2 - 4$$

Either by guessing or by rational root thm: $\lambda = 1$ is a root

$$\lambda^3 + 3\lambda^2 - 4 = (\lambda - 1)(\ \lambda^2 + \quad \lambda + \quad) \ ?$$

$$\Rightarrow \lambda^3 + 3\lambda^2 - 4 = (\lambda-1)(\lambda^2 + 4\lambda + 4) = (\lambda-1)(\lambda+2)^2$$

$$\Rightarrow \sigma(A) = \{1, -2\}$$

2.) Compute the eigenspace for each eigenvalue:

- $1 \cdot I_3 - A = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix}$

  Finding the row echelon form of and finding the kernel yields:

  $\Rightarrow \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis of $E_1(A) = \operatorname{Ker}(1 \cdot I_3 - A)$

- $(-2) \cdot I_3 - A = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix}$

  Finding the row echelon form of and finding the kernel yields:

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis of $E_{-2}(A) = \mathrm{Ker}\left((-2)\cdot I_3 - A\right)$

$\Rightarrow$ A has two eigenvalues but it has three indep. eigenvectors

and hence an eigenbasis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

$\Rightarrow \quad D = \begin{pmatrix} 1 & & \\ & -2 & \\ & & -2 \end{pmatrix} = P^{-1} A P$ with $P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

Thm. 5.10 Let $A \in \mathbb{R}^{n \times n}$ and $\sigma(A) = \{\underbrace{\lambda_1 \dots \lambda_k}_{\text{distinct list}}\}$, $k \leq n$

(1) $\mathrm{multgeom}_A(\lambda) \leq \mathrm{multalg}_A(\lambda) \quad \forall \lambda \in \sigma(A)$

(2) If A is diagonalizable then:

$$\mathrm{multgeom}_A(\lambda) = \mathrm{multalg}_A(\lambda)$$

(3) A diagonalizable $\iff \sum_{i=1}^{k} \mathrm{multgeom}_A(\lambda_i) = n$

Proof: (1) omitted.

(2) and (3) : HW11.