MATH-111(en) FALL 2024
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SOLUTIONS for Homework 7

Ex 7.1 (Non-subspaces of the plane)
Show that none of the following sets is a subspace of $\mathbb{R}^2$:

a) $S_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$;

b) $S_2 = \{(x, y) \in \mathbb{R}^2 : x \cdot y \geq 0\}$;

c) $S_3 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

(By the way, can you tell what each of these sets looks like? Try to draw them!)

Solution:
a) Consider $(x, y) = (1, 0) \in S_1$ and $\lambda = -1$. Then $\lambda(x, y) = (-1, 0) \notin S_1$ and hence S_1 is not a vector space.
Note that $(0, 0) \in S_1$ and $v + w \in S_1$ for all $v, w \in S_1$, so that is the only subspace axiom that property fails.

b) Consider $(x, y) = (1, 0) \in S_2$ and $(s, t) = (0, -1) \in S_2$. Then $(x, y) + (s, t) = (1, -1) \notin S_2$, and hence S_2 is not a vector space.
Again the other two subspace axioms are satisfied by S_2.

c) Consider the point $(x, y) = (1, 0)$ and $\lambda = 2$. Then $\lambda(x, y) = (2, 0)$, but $2^2 = 4 > 1$, so that $\lambda(x, y) \notin S_3$. Hence S_3 is not a vector space.
Alternative, you could argue that $(x, y) = (1, 0)$ and $(x', y') = (0, 1)$ are both in S_3 but their sum is not.

Drawings: S_1 is the first quadrant of the coordinate system (including non-negative parts of the x-axis and y-axis).
S_2 is the first and third quadrant of the coordinate system (including the x-axis and y-axis).
S_3 is the closed unit disk (that is, the disk with center 0 and radius 1 including its boundary circle)

Ex 7.2 (Is it a vector space?)
For each of the following sets (equipped with the obvious addition and scalar multiplication), decide whether it is a vector space and prove your result.

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = 0 \right\}, \quad B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y = 1 \right\}, \quad C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : z = y \right\}$$

$$D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \{0, -1, 1\} \right\}, \quad E = \big\{ f : \mathbb{R}^3 \to \mathbb{R}^3 \text{ linear} : f(e_1) = 0 \big\}$$

Solution:

- A is a vector space. We have that A is a subspace of $\mathbb{R}^3$. Indeed, it is clear that $0 \in A$. Moreover, if $x, y \in A$, then

$$(x+y)_1 = x_1 + y_1 = 0 + 0 = 0$$

implying that $x + y \in A$. Finally, if $\lambda \in \mathbb{R}$ and $x \in A$, then

$$(\lambda x)_1 = \lambda x_1 = \lambda \cdot 0 = 0$$

implying $\lambda x \in A$. Now by Lemma 4.2, any subspace of a vector space is itself a vector space, thus A is a vector space.

- B is not a vector space since it does not contain 0.

- C is a vector space. Again, by Lemma 4.2, it suffices to show C is a subspace of $\mathbb{R}^3$. It is clear that $0 \in C$. If $x, y \in C$, then

$$(x+y)_2 = x_2 + y_2 = x_3 + y_3 = (x+y)_3$$

which implies $x + y \in C$. Finally, if $\lambda \in \mathbb{R}$ and $x \in C$, then

$$(\lambda x)_2 = \lambda x_2 = \lambda x_3 = (\lambda x)_3$$

implying that $\lambda x \in C$. Thus, C is a subspace of $\mathbb{R}^3$ as required.

- D is not a vector space since

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in D, \text{ but } 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin D$$

which violates subspace axiom (S1).

Alternatively, you could argue as follows: The set D only contains 27 elements (as each of the three coordinate of a vector has three possible values, so there are 3^3 elements). Hence it is a finite subset of $\mathbb{R}^3$ and by Exercise 7.6, the only possible finite subspace has 1 element (which is the zero element).

- E is a vector space. We've seen that the space of linear functions

$$V = \{f : \mathbb{R}^3 \to \mathbb{R}^3 : f \text{ linear}\}$$

equipped with point-wise addition and point-wise scalar multiplication forms a vector space. Thus, by Lemma 4.2, it suffices to show that E is a subspace of V. $0 \in E$ as the zero map maps e_1 to 0. If $f, g \in E$, then

$$(f+g)(e_1) = f(e_1) + g(e_1) = 0 + 0 = 0$$

and thus, $f + g \in E$. Finally, if $f \in E$ and $\lambda \in \mathbb{R}$, then

$$(\lambda f)(e_1) = \lambda f(e_1) = \lambda \cdot 0 = 0$$

implying $\lambda f \in E$. Consequently, E is a subspace of V as claimed.

Ex 7.3 (Spaces of polynomials)

Let P_n be the vector space of polynomials of degree less than or equal to n. Determine if each of the following sets is a subspace of P_n for a given n. (You may take for granted that P_n is a vector space.)

a) The set of polynomials of the form $p(t) = at^2$ where a is an arbitrary real number.

b) The set of polynomials of the form $p(t) = a + t^2$ where a is an arbitrary real number.

c) The set of polynomials of the form $p(t) = c_1t^3 + c_2t^2 + c_3t + c_4$, where c_1, c_2, c_3 and c_4 are non-negative integers.

d) The set of polynomials in P_n that satisfy $p(0) = 0$.

Solution:

a) The set of polynomials of the form $p(t) = at^2$ is the set spanned by the element t^2 of P_2; thus is it a subspace by Theorem 4.3.

b) The set of polynomials of the form $p(t) = a + t^2$ is not a subspace because it does not contain the zero polynomial.

c) The set of polynomials with integer coefficients is not a sub-space. This is because multiplying one of its element with $\sqrt{2} \in \mathbb{R}$, we get a polynomial whose coefficients are not integers.

d) The set of polynomials of degree less than or equal to n such that $p(0) = 0$ is a subspace of P_n because

   - the zero polynomial clearly belongs to this set,
   - the sum of two polynomials which are zero when evaluated in zero is also zero when evaluated in zero,
   - if $p(0) = 0$, then $\lambda p(0) = 0$ for λ an arbitrary real number.

Ex 7.4 (Sum of subspaces)

Let V be a vector space and let S and T be subspaces of V. Prove that:

(a) $S \cap T$ is a subspace of V

(b) $S + T$ is a subspace of V

(c) $S \cup T$ is not a subspace of V.

Solution:

a) We show that $S \cap T$ satisfy the desired properties:

- Since S and T are both subspaces, $0 \in S$ and $0 \in T$ and so, $0 \in S \cap T$.
- If $x, y \in S \cap T$, it follows that $x, y \in S$ and so, $x + y \in S$. Similarly, $x, y \in S \cap T$ also implies that $x, y \in T$ and so, $x + y \in T$. Thus, by definition of the intersection, $x + y \in S \cap T$.

- Let $\lambda \in \mathbb{R}$ and $x \in S \cap T$. Then, as $x \in S$ implies $\lambda x \in S$ and similarly, $x \in T$ implies $\lambda x \in T$, it follows that $\lambda x \in S \cap T$ as required.

b) We show that $S + T$ satisfy the desired properties

- Since S and T are both subspaces, they both contain the zero vector 0 of V, hence $S+T$ contains $0 + 0 = 0$.

- If $s + t$ and $s' + t'$ are elements of $S + T$, then $(s + t) + (s' + t') = (s + s') + (t + t')$ is an element of $S + T$, because $s + s' \in S$ and $t + t' \in T$.

- If $s+t$ is an element of $S+T$, and $\lambda \in \mathbb{R}$, then $\lambda \cdot (s+t) = (\lambda s)+(\lambda t)$ is also an element of $S + T$, because $\lambda s \in S$ and $\lambda t \in T$.

c) For this, we construct a counter example: Let $V = \mathbb{R}^2$ and let S be the span of e_1 and T be the span of e_2. They are both subspaces by Theorem 4.3. Now, by observing that $e_1, e_2 \in S \cup T$ while $e_1 + e_2 \notin S \cup T$ since neither S nor T contains $e_1 + e_2$, it follows that $S \cup T$ is not a subspace.

Ex 7.5 (A subspace of polynomials)
Let $\mathbb{P}_3$ be the vector space of polynomials $p(t)$ of degree at most 3.

(a) Let S be the subspace spanned by

$$p_1(t) = 1 + t^2, \quad p_2(t) = 3t + 4t^3, \quad p_3(t) = 1 + t + 5t^2 + 4t^3.$$

Is $1 + 2t + 3t^2 + 4t^3$ an element of S?

(b) Define $\tilde{\mathbb{P}_3}$ to be the set of polynomials of degree exactly 3. Is $\tilde{\mathbb{P}_3}$ a vector space?

Solution:

a) We need to check whether there exist real numbers c_1, c_2, c_3 such that

$$c_1p_1(t) + c_2p_2(t) + c_3p_3(t) = 1 + 2t + 3t^2 + 4t^3.$$

Let us write the left-hand side explicitly collecting the coefficients in from of each monomial, i.e.,

$$(c_1 + c_3) \cdot 1 + (3c_2 + c_3)t + (c_1 + 5c_3)t^2 + (4c_2 + 4c_3)t^3 = 1 + 2t + 3t^2 + 4t^3.$$

As the monomials are linearly independent, this leads to the condition

$$c_1 + c_3 = 1, \qquad 3c_2 + c_3 = 2, \qquad c_1 + 5c_3 = 3, \qquad 4c_2 + 4c_3 = 4.$$

This is a system of linear equations that we can solve with row reduction as follows:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 1 & 0 & 5 & 3 \\ 0 & 4 & 4 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & 4 & 4 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 4 & 2 \\ 0 & 4 & 4 & 4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{8}{3} & \frac{4}{3} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From the echelon form, we see that the system has a solution, so we know that the polynomial is indeed an element of S. The coefficients are $c_1 = c_2 = c_3 = \frac{1}{2}$.
b) No since it does not contain the zero polynomial.

Ex 7.6 (The only finite subspaces is $\{0_v\}$.)
Let V be a vector space and 0_V its zero element. Prove that $\{0_V\}$ is the only subspace of V that consists of only finitely many elements.

Solution:

Let H be a subspace of V and suppose that it is not $\{0_V\}$. We will show that it contains infinitely many elements.
Since H is a subspace, $\{0_V\} \subseteq H$ and thus, $H \neq \{0_V\}$ implies that there exists some $x \in H$ where $x \neq 0_V$.
Moreover, H is a subspace implies that, for all $k \in \mathbb{R}$: $kx \in H$.
We will prove the following claim:

Claim: for each k, kx is a different element of H.

To prove Ex.7.6, it suffices to prove the claim because $\mathbb{R}$ has infinitely many elements and so there are infinitely many different elements of H that have the form kx.

Proof of claim: Let $k_1 \neq k_2$ be two numbers in $\mathbb{R}$. Assume for a contradiction that $k_1 x = k_2 x$. Then, it follows that $(k_1 - k_2)x = 0$.
By assumption $k_1 - k_2 \neq 0$. So we can divide both side of the equation by $k_1 - k_2$.
This implies $x = 0$ which contradicts the fact that $x \neq 0$.

Ex 7.7 (Column space and kernel)

(a) Does $v = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ lie in the column space of $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{pmatrix}$? Does it lie in its kernel?

(b) Let $B = \begin{pmatrix} 1 & 2 & -3 \\ 4 & -1 & 0 \\ 0 & -3 & 4 \end{pmatrix}$. Find a nonzero vector $u \in \mathrm{Col}(B)$ and a nonzero vector $v \in \mathrm{Ker}(B)$. Is there a nonzero vector that lies in both $\mathrm{Col}(B)$ and $\mathrm{Ker}(B)$?

(c) Express the kernel of the following matrix in parametric vector form:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

Solution:

(a) We want to know if there is a solution to the system $Ax = v$, so we solve this system by row reduction:

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & -2 \\ -3 & 4 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 0 & -2 \\ 0 & 1 & 1 & 1 \\ -3 & 4 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & -1 \\ 0 & 1 & 1 & 1 \\ -3 & 4 & 1 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & \frac{11}{2} & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{9}{2} & -\frac{13}{2} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{13}{9} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & -1 \\ 0 & 1 & 0 & -\frac{4}{9} \\ 0 & 0 & 1 & \frac{13}{9} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{7}{9} \\ 0 & 1 & 0 & -\frac{4}{9} \\ 0 & 0 & 1 & \frac{13}{9} \end{array}\right)$$

And indeed:

$$v = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = -\frac{7}{9}\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} - \frac{4}{9}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + \frac{13}{9}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

so v lies in $\mathrm{Col}(A)$. Of course, we could have stopped the row reduction the moment we saw that there were three non-zero entries on the diagonal.
To see if v is in the kernel of A, we simply calculate:

$$Av = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -9 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so it is not.

(b) Finding a nonzero vector in the column space is easy, we can just take any column of the matrix. To find a column in the kernel we'll have to do row reduction on the system $Bx = 0$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -9 & 12 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So all solutions are given by $x = t/3$, $y = 4t/3$, $z = t$, and for instance $t = 3$ gives the nonzero vector $v = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ in the kernel
To find a vector that is in both, note that all vectors in the kernel have the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \cdot v = s \cdot \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

So if any of these is in the column space, then so is v. Therefore we can answer this by seeing if the system $Bx = v$ has a solution, using row reduction:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 4 & -1 & 0 & 4 \\ 0 & -3 & 4 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & -9 & 12 & 0 \\ 0 & -3 & 4 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & -3 & 4 & 0 \end{array}\right)$$

So this system has no solution, hence there is no nonzero vector that is in both the column space and the kernel.

(c) We do row reduction:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 2 & 2 & 2 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the reduced echelon form, so the solution set of $Ax = 0$ is $x_1 = -x_3$ and $x_2 = -x_4$. Introducing s and t as parameters for x_3 and x_4, we get that the parametric vector form of the solution set is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ -t \\ s \\ t \end{pmatrix} = s \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Ex 7.8 (Column space and kernel)
(a) Consider

$$w = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 & -5/2 \\ -3 & -2 & 4 \\ 2 & 4 & -4 \end{pmatrix}.$$

Find out if w is in $\operatorname{Col} A$, in $\operatorname{Ker} A$, or both.

(b) Find bases for the kernel, the column space, and the row space of $A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 2 & 4 & 14 & 4 \\ 2 & 3 & 12 & 3 \end{pmatrix}$

Solution:

(a)
w is in $\operatorname{Ker} A$ because $Aw = 0$. w is in $\operatorname{Col} A$ because the system $Ax = w$ is consistent. (You can have a look at the reduced echelon form of its augmented matrix. A possible solution of $Ax = w$ is $x = (-1, 1, 0)$.)

(b)
Follow the steps of Example ($\star$) from class (Week 7). You will find that the reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In conclusion, you get the following bases for the kernel, row space, and column space of A:

$$\mathcal{B}_{\operatorname{Ker}(A)} = \left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \mathcal{B}_{\operatorname{Col}(A)} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}, \quad \mathcal{B}_{\operatorname{Row}(A)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\},$$

Please be aware that different solutions are possible as no vector space has a unique basis. (If you followed the steps from Example ($\star$) from class, the bases you obtain are exacly as above.) No matter what the elements of the bases in our own solutions are (e.g. if they are different from the above solution), the number of elements in each basis must be exactly as in the above solution.
If you need a repetition, the following youtube video goes over this exact example:
`https://www.youtube.com/watch?v=AVWTlNkTNfw`

Ex 7.9 ((When) do linear maps preserve linear (in)dependence?)
Let V and W be two vector spaces, $T : V \to W$ a linear transformation and $\{v_1, \ldots, v_p\}$ a subset of V.

1. Show that if the set $\{v_1, \ldots, v_p\}$ is linearly dependent, then the set $\{T(v_1), \ldots, T(v_p)\}$ is linearly dependent too.
2. Assume T is an injective transformation : $T(u) = T(v) \Rightarrow u = v$. Show that if the set $\{T(v_1), \ldots, T(v_p)\}$ is linearly dependent, then the set $\{v_1, \ldots, v_p\}$ is linearly dependent too.

Solution:

1. If $\{v_1, \ldots, v_p\}$ are linearly dependent, then there exist $c_1, \ldots, c_p$, such that not all c_i's are zero and
$$c_1 v_1 + \ldots + c_p v_p = 0.$$
As T is linear,
$$T(c_1 v_1 + \ldots + c_p v_p) = T(0) = 0$$
and hence
$$c_1 T(v_1) + \ldots + c_p T(v_p) = T(c_1 v_1 + \ldots + c_p v_p) = 0.$$
As not all c_i's are zero, $\{T(v_1), \ldots, T(v_p)\}$ are linearly dependent.
2. We prove the statement by proving its contraposition, i.e., that if $\{T(v_1), \ldots, T(v_p)\}$ are linearly dependent, then so are $\{v_1, \ldots, v_p\}$.
If $\{T(v_1), \ldots, T(v_p)\}$ are linearly dependent, then there exist $c_1, \ldots, c_p$, such that not all c_i's are zero and
$$c_1 T(v_1) + \ldots + c_p T(v_p) = 0.$$
As T is linear, $0 = T(0)$. Hence
$$T(c_1 v_1 + \ldots + c_p v_p) = c_1 T(v_1) + \ldots + c_p T(v_p) = 0 = T(0).$$
As T is injective by assumption, this equation implies that $c_1 v_1 + \ldots + c_p v_p = 0$, which shows that the vectors $\{v_1, \ldots, v_p\}$ are linearly dependent.

Ex 7.10 (A subspace and a possible basis)
Let $S \subset \mathbb{R}^4$ be the subset of vectors $(x_1, x_2, x_3, x_4)^T$ satisfying the equations
$$x_1 - 2x_3 + x_4 = 0, \quad x_2 + 3x_3 = 0, \quad \text{and} \quad x_1 - x_4 = 0.$$
Show that S is a subspace of $\mathbb{R}^4$. Find a basis for S.

Solution:
We check the three conditions for being a subspace. The zero vector is in S because it satisfies those equations. If (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) satisfy those equations, then so do $(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$ and (cx_1, cx_2, cx_3, cx_4), because for instance
$$(x_1 + y_1) - 2(x_3 + y_3) + (x_4 + y_4) = (x_1 - 2x_3 + x_4) + (y_1 - 2y_2 + y_4) = 0 + 0 = 0,$$
$$(cx_1) - 2(cx_3) + (cx_4) = c \cdot (x_1 - 2x_3 + x_4) = c \cdot 0 = 0,$$
and the same for the other two equations.

Alternatively, S can be described as the kernel of the matrix

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Therefore, in order to find a basis, we can use row reduction to find all solutions of the following system of equations:

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$$\Longrightarrow \quad x_4 = t \text{ is a free variable}, \quad x_1 = t, \quad x_2 = -3t, \quad x_3 = t.$$

We can write the solutions in vector form like so:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t \\ -3t \\ t \\ t \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ -3 \\ 1 \\ 1 \end{pmatrix}.$$

This means that

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is a basis for S, because it spans S (every element of S is a linear combination of elements of the basis, which in this case is just a multiple of that one vector), and it is linearly independent (containing a single nonzero vector).

Ex 7.11 (The range of linear maps)
Let V, W be vector spaces and $T : V \to W$ be linear. Show that $\mathrm{Ran}(T)$ is a subspace of W.

Solution:
By linearity, we have that $T(0_V) = T(0 \cdot 0_V) = 0 \cdot T(0_V) = 0_W$, so that $0_W \in \mathrm{Ran}(T)$. Moreover, if $w_1, w_2 \in \mathrm{Ran}(T)$, then by definition there exist $v_1, v_2 \in V$ such that $w_1 = T(v_1)$ and $w_2 = T(v_2)$. Then by linearity of T we have $T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$, so that $w_1 + w_2 \in \mathrm{Ran}(T)$. Finally, for $\lambda \in \mathbb{R}$ we have that $T(\lambda v_1) = \lambda T(v_1) = \lambda w_1$, so that also $\lambda w_1 \in \mathrm{Ran}(T)$. Thus $\mathrm{Ran}(T)$ satisfies all properties of a subspace of W.

Ex 7.12 (True/False questions)

Decide whether the following statements are always true or if they can be false.

(i) Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Then the set of functions such that $f(3) = 0$ is a subspace.

(ii) Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Then the set of functions such that $f(3) \cdot f(6) = 0$ is a subspace.

(iii) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and let S be the subset of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a, b \in \mathbb{R}$. Then S is a subspace.

(iv) Let $A \in \mathbb{R}^{n\times n}$. If $\mathrm{Ker}(A) = \{0\}$ then $\mathrm{Ker}(A^2) = \{0\}$.

(v) Let $A \in \mathbb{R}^{n\times n}$. Then $\mathrm{Ker}(A) = \{0\}$ if and only if $\mathrm{Ran}(A) = \mathbb{R}^n$.

(vi) Let $A \in \mathbb{R}^{n\times n}$. Then $\mathrm{Ker}(A) = \mathbb{R}^n$ if and only if $\mathrm{Ran}(A) = \{0\}$.

Solution:

(i) **True:** The zero function ($f(x) = 0$ for all $x \in \mathbb{R}$) satisfies the condition. If f and g are in V, then so is $f+g$ because $(f+g)(3) = f(3) + g(3) = 0 + 0 = 0$. If $f(3) = 0$ and $c \in \mathbb{R}$ then $(cf)(3) = c \cdot f(3) = c \cdot 0 = 0$.

(ii) **False:** The zero vector is there and V contains scalar multiples, but it does not contain all sums. For instance, let $f_1(3) = 1$ and $f_1(t) = 0$ for all $t \neq 3$, and let $f_2(6) = 1$ and $f_2(t) = 0$ for all $t \neq 6$. Then $f_1(3) \cdot f_1(6) = 1 \cdot 0 = 0$ and $f_2(3) \cdot f_2(6) = 0 \cdot 1 = 0$, so both are in this set. But $f_1 + f_2$ has

$$(f_1 + f_2)(3) \cdot (f_1 + f_2)(6) = (f_1(3) + f_2(3)) \cdot (f_1(6) + f_2(6)) = (1+0) \cdot (0+1) = 1,$$

so is not in this set.

(iii) **False:** It is not a subspace, for instance because it does not contain the zero matrix, or because

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ 0 & 2 \end{pmatrix} \notin S$$

because of the wrong entry at position $(2,2)$.

(iv) **True:** Let $x \in \mathrm{Ker}(A^2)$. We want to prove that x must be the zero vector.
$x \in \mathrm{Ker}(A^2)$ means that $A^2 x = 0$.
This can also be written as $A(Ax) = 0$. Hence $Ax \in \mathrm{Ker}(A)$.
So by our assumption that $\mathrm{Ker}(A) = \{0\}$, it follows that $Ax = 0$.
But this means that $x \in \mathrm{Ker}(A)$. Again, by the same assumption, this implies that $x = 0$.

(v) **True:** $\mathrm{Ker}(A) = \{0\}$ means that the linear transformation $f(x) = Ax$ is injective. Since A is square, by the alphabet Theorem (Thm.2.7) it follows that f is surjective. But this means $\mathrm{Ran}(f) = \mathbb{R}^n$, which is equivalent to $\mathrm{Ran}(A) = \mathbb{R}^n$.

(vi) **True:** $\mathrm{Ker}(A) = \mathbb{R}^n$ means that for every $x \in \mathbb{R}^n$, $Ax = 0$. This means that 0 is the only element in $\mathrm{Ran}(A)$.