MATH-111(en) FALL 2024
Linear Algebra Annina Iseli

SOLUTIONS for Homework 6

Ex 6.1 (The determinant of triangular matrices)

Let $A \in \mathbb{R}^{n\times n}$ be a lower (resp. upper) triangular matrix. Show that the determinant of A equals the product of its diagonal elements, i.e., $\det(A) = a_{11} \cdot \ldots \cdot a_{nn} = \prod_{i=1}^{n} a_{ii}$.
Hint: Lower triangular A: use the definition of the determinant and induction on n. Upper triangular A: use that $\det(A^T) = \det(A)$.

Solution:
As suggested in the hint, we start considering an lower triangular matrix A, that is, its coefficients satisfy $a_{ij} = 0$ whenever $j > i$. For $n = 1$, the statement is true. Now assume that it holds for matrices of size $(n-1)\times(n-1)$. Since $a_{1j} = 0$ for all $j \geq 2$, the definition of the determinant yields

$$\det(A) = a_{11}\det(A_{11}).$$

Since A_{11} is obtained from A by deleting the first row and the first column, the matrix A_{11} is still a lower triangular matrix (formal argument: the coefficient $(A_{11})_{ij}$ is given by $a_{(i+1)(j+1)}$, so that $j > i$ implies $j+1 > i+1$ and therefore $(A_{11})_{ij} = 0$ for such indices; but reasoning by intuition suffices in this case). Moreover, the diagonal elements of A_{11} are given by $a_{22}, \ldots, a_{nn}$ and therefore the induction hypothesis gives

$$\det(A) = a_{11} \cdot \prod_{i=2}^{n} a_{ii} = \prod_{i=1}^{n} a_{ii}.$$

When A is an upper triangular matrix, then A^T is a lower triangular matrix with the same diagonal elements. Hence the claim follows from $\det(A) = \det(A^T)$ and the result for lower triangular matrices.

Ex 6.2 (Row reduction and the determinant)

Let $A \in \mathbb{R}^{n\times n}$ and assume that A_1 is obtained from A by applying an elementary operation on the rows or columns of A. Show that

$$\det(A) = \begin{cases} -\det(A_1) & \text{if two rows or columns have been swapped,} \\ \lambda^{-1}\det(A_1) & \text{if a row or column has been multiplied by } \lambda \neq 0, \\ \det(A_1) & \text{if a multiple of a row (resp. column) has been added to another row (resp. colu} \end{cases}$$

Hint: We know that $A_1 = EA$ for some elementary matrix E.

Solution:
Using the equation $A_1 = EA$ and the multiplicativity of the determinant, we know that $\det(A_1) = \det(E)\det(A)$, while the determinant of elementary matrices is given by Lemma 3.3. In particular, we have

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ swaps two rows,} \\ \lambda & \text{if } E \text{ multiplies a row by } \lambda \neq 0, \\ 1 & \text{if } E \text{ adds a multiple of a row to another row.} \end{cases}$$

Thus the claim follows by noticing that $\det(A) = \det(A_1)/\det(E)$. The respective statements for columns follow by either repeating the same argument for $A_1 = EA$, or, by taking transposes.

Ex 6.3 (Different methods for computing determinants)
Compute the determinant of each of the following matrices in three ways: Once using cofactor expansion across a row, once using cofactor across a column, and once with row reduction.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 8 & 0 \end{pmatrix}$$

Solution:
We expand A over the first row:

$$\det(A) = 2 \cdot \det(A_{11}) - 0 \cdot \det(A_{12}) + 0 \cdot \det(A_{13}) = 2 \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 2 \cdot (1 \cdot 4 - 2 \cdot 3) = -4;$$

Next we expand A over the first column:

$$\det(A) = 2 \cdot \det(A_{11}) - 2 \cdot \det(A_{21}) + 0 \cdot \det(A_{31}) = 2 \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 2 \det \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = -4;$$

Finally, we use row reduction. Because here we only use the operation that adds a multiple of a row to another row, we get the same determinant everywhere, and we don't have to multiply by any factor.

$$\det(A) = \det \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix} = \det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix} = \det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix} = 2 \cdot 1 \cdot (-2) = -4.$$

When doing cofactor expansion, we can choose what row or column to do it over, and often it's easiest to pick a row or column with a lot of zeros, since then the expansion will have fewer terms. For B, it is easiest to expand over the third row and third column. (Of course, it isn't wrong to use a different row or column.)

$$\det(B) = 0 \cdot \det(A_{31}) - 8 \cdot \det(A_{32}) + 0 \cdot \det(A_{33}) = -8 \cdot \det \begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix} = -8 \cdot (1 \cdot 0 - 3 \cdot 4) = 96;$$

$$\det(B) = 3 \cdot \det(A_{13}) - 0 \cdot \det(A_{23}) + 0 \cdot \det(A_{33}) = 3 \cdot \det \begin{pmatrix} 4 & 5 \\ 0 & 8 \end{pmatrix} = 3 \cdot (4 \cdot 8 - 5 \cdot 0) = 96;$$

When row reducing B, it is convenient to divide the second row by -3, so we have to multiply the determinant by -3 (or in other words, we multiply the row by $1/(-3)$, so we have to multiply the determinant by $1/(1/(-3)) = -3$).

$$\det(B) = \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 8 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -12 \\ 0 & 8 & 0 \end{pmatrix} = -3 \cdot \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 8 & 0 \end{pmatrix}$$

$$= -3 \cdot \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -32 \end{pmatrix} = -3 \cdot 1 \cdot 1 \cdot (-32) = 96.$$

We can also do it with only replacement, by adding 8/3 times the second row to the third row:

$$\det(B) = \det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 8 & 0 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -12 \\ 0 & 8 & 0 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -12 \\ 0 & 0 & -32 \end{pmatrix} = 1 \cdot (-3) \cdot (-32) = 96.$$

Ex 6.4 (Determinants based on another determinant)
Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and assume that $\det(A) = 7$. Compute the determinants of the following matrices

$$B = \begin{pmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{pmatrix}, \quad C = \begin{pmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{pmatrix}.$$

Solution:
$\det(B) = 7$. Indeed, we obtained this matrix from the original matrix by adding the second row to the first row, which does not change the value of the determinant.
$\det(C) = 14$. Indeed, we obtained this matrix from the original one by multiplying the second row by 2, which multiplies the determinant by 2, then by adding the first row to the second, which does not change the value of the determinant.

Ex 6.5 (More determinants)
Compute the determinants of the following matrices (You may use your preferred method or try to practice different methods.)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 10 & 5 & 10 & 5 \\ 6 & 9 & 0 & -3 \\ 3 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Solution:
For the first matrix, let's use cofactor expansion over the second row (note that the sign in front of the cofactor of a_{21} is $(-1)^{2+1} = -1$):

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix} = -1 \cdot \det\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = -(2 \cdot 1 - 3 \cdot 1) = 1.$$

For the second matrix, let's use row reduction:

$$\det\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix} = -\det\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = -\det\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & \frac{2}{3} \end{pmatrix} = -\left(1 \cdot 1 \cdot (-3) \cdot \frac{2}{3}\right) = 2.$$

For the third matrix we can use a combination of row reduction and cofactor expansion (this is often a good combination): With row reduction we create a row or column with a single nonzero entry, then we expand on that row or column.

$$\det\begin{pmatrix}10&5&10&5\\6&9&0&-3\\3&0&0&3\\1&0&1&1\end{pmatrix} = 5{\cdot}3{\cdot}3{\cdot}\det\begin{pmatrix}2&1&2&1\\2&3&0&-1\\1&0&0&1\\1&0&1&1\end{pmatrix} = 45\det\begin{pmatrix}2&1&2&1\\2&3&0&-1\\1&0&0&1\\0&0&1&0\end{pmatrix} = -45\det\begin{pmatrix}2&1&1\\2&3&-1\\1&0&1\end{pmatrix}$$

$$= -45\det\begin{pmatrix}0&1&-1\\0&3&-3\\1&0&1\end{pmatrix} = -45\det\begin{pmatrix}1&-1\\3&-3\end{pmatrix} = -45(1\cdot(-3)-(-1)\cdot 3) = 0.$$

Ex 6.6 (Determinant of an antidiagonal matrix)
Find the determinant of the following *antidiagonal* matrix:

$$\begin{pmatrix}0&0&0&a\\0&0&b&0\\0&c&0&0\\d&0&0&0\end{pmatrix}$$

What is the determinant of an $n\times n$ antidiagonal matrix with all its antidiagonal entries equal to 2?

Solution:

$$\det\begin{pmatrix}0&0&0&a\\0&0&b&0\\0&c&0&0\\d&0&0&0\end{pmatrix} = -\det\begin{pmatrix}d&0&0&0\\0&0&b&0\\0&c&0&0\\0&0&0&a\end{pmatrix} = (-1)^2\det\begin{pmatrix}d&0&0&0\\0&c&0&0\\0&0&b&0\\0&0&0&a\end{pmatrix} = (-1)^2abcd = abcd.$$

For an $n\times n$ antidiagonal matrix with all its antidiagonal entries equal to 2, if we can write $n=2k$ or $n=2k+1$, then the determinant is $(-1)^k2^n$. This is because we can use k row interchanges to make it diagonal: We first interchange the first and n-th row, then the second and $(n-1)$-th row, etc., similar to in the example. If $n=2k$ is even, we will be done after k steps, and if $n=2k+1$ is odd, then we are also done after k steps, because the middle entry can stay where it is.

Ex 6.7 (Determinants and volume)

(a) Calculate the volume of the parallelepiped with the following vertices:

$$\begin{pmatrix}0\\0\\0\end{pmatrix},\quad\begin{pmatrix}1\\4\\0\end{pmatrix},\quad\begin{pmatrix}-2\\-5\\2\end{pmatrix},\quad\begin{pmatrix}-1\\2\\-1\end{pmatrix},\quad\begin{pmatrix}-1\\-1\\2\end{pmatrix},\quad\begin{pmatrix}0\\6\\-1\end{pmatrix},\quad\begin{pmatrix}-3\\-3\\1\end{pmatrix},\quad\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

(b) Calculate the area of the triangle whose vertices are the points $(1,2),(2,4),(3,3)\in\mathbb{R}^2$.

Solution:

(a) We first have to determine which vertices determine the parallelepiped. If we call the second vector $\mathbf{u}$, the third $\mathbf{v}$, and the fourth $\mathbf{w}$, then $\mathbf{u}+\mathbf{v}$ is the fifth vector, $\mathbf{u}+\mathbf{w}$ is the sixth,

$\mathbf{v}+\mathbf{w}$ is the seventh, and finally $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is the eighth. That means that the vertices $\mathbf{u}, \mathbf{v}, \mathbf{w}$ determine the parallelepiped.

Then the volume of the parallelepiped is the absolute value of the determinant of the matrix with these vectors for columns (in any order, because changing the order won't change the absolute value).

$$\text{volume} = \left|\det\begin{pmatrix}1 & -2 & -1\\ 4 & -5 & 2\\ 0 & 2 & -1\end{pmatrix}\right| = \left|\det\begin{pmatrix}1 & -2 & -1\\ 0 & 3 & 6\\ 0 & 2 & -1\end{pmatrix}\right|$$
$$= \left|\det\begin{pmatrix}1 & -2 & -1\\ 0 & 3 & 6\\ 0 & 0 & -5\end{pmatrix}\right| = |1\cdot 3\cdot(-5)| = 15.$$

(b) This triangle has the same area as the triangle spanned by $\begin{pmatrix}0\\0\end{pmatrix}$ and the two vectors $\begin{pmatrix}3\\3\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$. Or, equivalently, after a translation by $-\begin{pmatrix}1\\2\end{pmatrix}$ we get the three vectors $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$, and their triangle must have the same area.

The area of that triangle is half the area of the parallelogram spanned by $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\end{pmatrix}$, which is the determinant of the matrix obtained by taking those two vectors as columns. So

$$\text{area of the triangle } = \frac{1}{2}\det\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = \frac{1}{2}(2\cdot 2 - 1\cdot 1) = \frac{3}{2}.$$

Ex 6.8 (Abstract determinant calculations)

Show that

(a) if A is an invertible matrix, then $\det(A^{-1}) = 1/\det A$,

(b) if A and P are square matrices, with P invertible, then $\det(PAP^{-1}) = \det A$,

(c) if U is a square matrix such that $U^T U = I$, then $\det U = \pm 1$,

(d) if A is a square matrix such that $\det(A^4) = 0$, then A cannot be invertible.

Solution:

(a) $(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1$, and thus $\det A^{-1} = 1/\det A$.

(b) $\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1}) = \det(PP^{-1})(\det A) = (\det I)(\det A) = \det A$.

(c) $1 = \det I = \det(U^T U) = (\det U^T)(\det U) = (\det U)^2$, and thus $\det U = \pm 1$.

(d) $0 = \det A^4 = (\det A)^4$, and thus $\det A = 0$, which implies that A is not invertible.

Ex 6.9 (Multiple choice and True/False questions)

a) Let A be an $n\times n$ matrix with nonzero determinant. Then $\det(A+A) =$
(A) 0 (B) $\det(A)$ (C) $2\det(A)$ (D) $2^n\det(A)$.

b) In the following, we assume that all the matrices involved are square matrices. Decide whether the following statements are always true or if they can be false.

(i) The following matrix is invertible.

$$\begin{pmatrix} 0 & 0 & 0 & 4 \\ 2 & 0 & 4 & 3 \\ 1 & 0 & 2 & 5 \\ 3 & 6 & 1 & 8 \end{pmatrix}$$

(ii) The (i, j)-cofactor of a square matrix A is the matrix A_{ij} obtained by deleting form A its i-th row and j-th column.

(iii) If A and B are both $n \times n$, then $\det(A + B) = \det(A) + \det(B)$.

(iv) If A and B are row equivalent, i.e., they can be obtained from each other by finitely many elementary operations, then they have the same determinant.

(v) The linear transformation associated to A is injective if and only if $\det(A) \neq 0$.

(vi) The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.

(vii) The determinant of A is the product of the diagonal entries in A.

(viii) If $\det A$ is zero, then two rows or two columns are the same, or a row a column is zero.

Solution:

a) **The answer is (D).** $A + A$ (or $2A$) is obtained from A by doubling each of the rows. We know that multiplying a row by a number c multiplies the determinant by c. Since we multiply each of n rows with 2, the determinant becomes $2^n \det(A)$.

b) (i) **False.** The determinant is easy to calculate by cofactor expansion over the first row, and then for the 3×3 determinant using cofactor expansion over the second column:

$$\det \begin{pmatrix} 0 & 0 & 0 & 4 \\ 2 & 0 & 4 & 3 \\ 1 & 0 & 2 & 5 \\ 3 & 6 & 1 & 8 \end{pmatrix} = 0 \cdot \det(A_{11}) - 0 \cdot \det(A_{12}) + 0 \cdot \det(A_{13}) - 4 \cdot \det(A_{14}) = -4 \cdot \det \begin{pmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ 3 & 6 & 1 \end{pmatrix}$$

$$= (-4) \cdot (-6) \cdot \det \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = (-4) \cdot (-6) \cdot (2 \cdot 2 - 4 \cdot 1) = 0.$$

So the matrix is not invertible.
We could of course also have done this with row reduction, but in this case the determinant is faster.

(ii) **False.** There is also the factor $(-1)^{i+j}$.

(iii) **False.** Take for instance $A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then $\det(A) + \det(B) = 0 + 0 = 0$, but $\det(A + B) = 1$.

(iv) **False.** If B is obtained from A by a row interchange, or by multiplying a row by a number other than 0 or 1, then they are row equivalent, but have different determinant.

(v) **True.** We saw that $\det(A) \neq 0$ if and only if A is invertible, and by the Invertible Matrix Theorem that we saw earlier, this is equivalent to the transformation defined by A being injective (if A is square).

(vi) **False.** There may be factors coming from multiplying a row by a scalar $\lambda \notin \{0, 1\}$.

(vii) **False.** For instance, if $bc \neq 0$, then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq ad.$$

(viii) **False.** Rows or columns can be linearly dependent without one of them being zero or two of them being equal to each other.