MATH-111(en) FALL 2024
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SOLUTIONS for Homework 13

Ex 13.1 (Using the Gram–Schmidt process)

Let W be the subspace of $\mathbb{R}^4$ spanned by the basis vectors

$$b_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix} \quad \text{and} \quad b_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

a) Construct an orthogonal basis for W using the Gram–Schmidt process.

b) Consider $A = [b_1 \quad b_2 \quad b_3]$ having the vectors b_1, b_2, b_3 as columns. Find out a QR decomposition of A.

Solution:

a) We can use the Gram–Schmidt process to construct an orthogonal basis $\{v_1, v_2, v_3\}$ as follows.

First we set $v_1 = b_1$. Then to find v_2, we subtract from b_2 its projection on the subspace W_1 spanned by $b_1 = v_1$. That is we compute:

$$\begin{aligned} v_2 &= b_2 - \mathrm{proj}_{W_1} b_2 = b_2 - \frac{b_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\ &= \begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \\ -3/2 \\ -3/2 \end{pmatrix}. \end{aligned}$$

As v_2 is the component of b_2 orthogonal to b_1, $\{v_1, v_2\}$ is an orthogonal basis of the subspace W_2 spanned by b_1 and b_2. The last step is to subtract from b_3 its projection on the subspace W_2 and dub this result v_3.

$$\begin{aligned} v_3 &= b_3 - \mathrm{proj}_{W_2} b_3 = b_3 - \frac{b_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{b_3 \cdot v_2}{v_2 \cdot v_2} \mathbf{v}_2 \\ &= \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 3/2 \\ 3/2 \\ -3/2 \\ -3/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}. \end{aligned}$$

b) We have seen that if Q is a matrix whose columns constitute an orthonormal basis of $\mathrm{Col}(A)$, then a QR decomposition of the form $A = Q \cdot R$ exists.

To find such a Q, we can normalize the orthogonal basis $\{v_1, v_2, v_3\}$ that we just obtained:

$$Q = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{10} \\ -1/2 & 1/2 & 2/\sqrt{10} \\ -1/2 & -1/2 & 1/\sqrt{10} \\ 1/2 & -1/2 & 2/\sqrt{10} \end{pmatrix}.$$

Now, as the columns of Q are orthonormal, we have $Q^T Q = I$ (this also holds when Q is not square), so R is necessarily of the form

$$R = Q^T Q R = Q^T A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & \sqrt{10} \end{pmatrix}.$$

Ex 13.2 (Finding an orthonormal basis)
Find an orthonormal basis for the span of the following vectors.

$$\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$$

Solution:
We use the Gram–Schmidt process:

$$v_1 = b_1 = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \quad v_2 = b_2 - \frac{b_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix} - \frac{-50}{50} \cdot \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

Then $\{v_1, v_2\}$ is an orthogonal basis. To get an orthonormal basis $\{u_1, u_2\}$, we normalize both vectors to get unit vectors:

$$u_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{50}} \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \quad u_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

Ex 13.3 (QR factorization)
Find a QR factorization for each of the following matrices:

$$A = \begin{pmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{pmatrix}$$

Solution:
<u>Factorization for A</u>: First, find an orthogonal basis for the column space using Gram-Schmidt:

$$v_1 = \begin{pmatrix} -2 \\ 5 \\ 2 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 7 \\ -2 \\ 6 \end{pmatrix} - \frac{49}{49} \begin{pmatrix} -2 \\ 5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -4 \\ 2 \end{pmatrix}$$

Then we normalize these vectors and put them as columns of Q:

$$Q = \frac{1}{7}\begin{pmatrix} -2 & 5 \\ 5 & 2 \\ 2 & -4 \\ 4 & 2 \end{pmatrix}$$

Then $Q^T Q = I$, so as in Exercise 13.1b), we get $A = QR$ if

$$R = Q^T A = \frac{1}{7}\begin{pmatrix} -2 & 5 & 2 & 4 \\ 5 & 2 & -4 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 7 \\ 0 & 7 \end{pmatrix}$$

<u>Factorization for B</u>: Applying the Gram-Schmidt process to the columns of B, we obtain the following orthogonal basis of $\operatorname{Col} B$:

$$\begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ 3 \\ -1 \end{pmatrix}.$$

We do have to normalize them, but this is easy because they all have the same length $\sqrt{12}$. We get

$$Q = \frac{1}{\sqrt{12}}\begin{pmatrix} -1 & 3 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{pmatrix}$$

And finally we compute R:

$$R = Q^T B = \frac{1}{\sqrt{12}}\begin{pmatrix} -1 & 3 & 1 & 1 \\ 3 & 1 & 1 & -1 \\ -1 & -1 & 3 & -1 \end{pmatrix}\begin{pmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 6 & -18 & 3 \\ 0 & 6 & 15 \\ 0 & 0 & 6 \end{pmatrix}$$

Ex 13.4 (Proof of Theorem 6.13)

Theorem 6.13 states as follows: For a matrix $A \in \mathbb{R}^{m\times n}$ the following statements are equivalent:

(i) For every $b \in \mathbb{R}^m$, the equation $Ax = b$ has a unique least square solution.

(ii) $A^T A$ is invertible

(iii) The columns of A are linearly independent.

Solution: <u>$(i) \Leftrightarrow (ii)$</u> By Theorem 6.12, the space of all least square solutions of $Ax = b$ equals the solution spaces of the equation $A^T Ax = Ab$. Since $A \in \mathbb{R}^{m\times n}$, we have that $A^T A \in \mathbb{R}^{n\times n}$. Thus, in particular $A^T A$ is square and the solution spaces of $A^T Ax = Ab$ consists of exactly one element if and only if $A^T A$ is invertible. In summery, $Ax = b$ has a unique least square solution if and only if $A^T A$ is invertible.
<u>$(i) \implies (iii)$</u> Suppose otherwise, namely that the columns of A are linearly dependent, i.e. there exists $\lambda_1, \dots, \lambda_m$ not all of which are 0 such that $\lambda_1 A_1 + \cdots + \lambda_m A_m = 0$ (denoting A_i for the i-th column of A. Then, taking the vector $x = (\lambda_1, \dots, \lambda_m)^T \in \mathbb{R}^m$,

$$Ax = \lambda_1 A_1 + \cdots + \lambda_m A_m = 0.$$

Thus, x is a solution to $Ax = 0$. However, as we know that $A0 = 0$ and by (i), the solution to $Ax = 0$ is unique, this must implies that $x = 0$ which contradicts the fact that not all of the λ_i are zero.
$\underline{(iii) \implies (ii)}$ Recall that a square matrix is invertible if and only if it has trivial kernel. Thus, it suffices to show that $\ker A^T A = \{0\}$. Suppose $x \in \ker A^T A$, i.e. $A^T Ax = 0$, then

$$0 = x^T(A^T Ax) = (x^T A^T)Ax = (Ax)^T(Ax) = \|Ax\|^2$$

implying $Ax = 0$. However, as the columns of A are linearly independent, as

$$0 = Ax = x_1 A_1 + \cdots + x_m A_m$$

it follows that $x = 0$ implying $\ker A^T A = \{0\}$ as desired.

Ex 13.5 (A least-squares problem)
Find all least-squares solution $x^\star$ of the system $Ax = b$ and their least square errors $\|Ax^\star - b\|$.

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

Solution:
We have to solve $A^T Ax^* = A^T b$. We compute

$$A^T A = \begin{pmatrix} 12 & 8 \\ 8 & 10 \end{pmatrix}, \quad A^T b = \begin{pmatrix} -24 \\ -2 \end{pmatrix}$$

and then row reduce

$$\left(\begin{array}{cc|c} 12 & 8 & -24 \\ 8 & 10 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 3 & 2 & -6 \\ 24 & 30 & -6 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 3 & 2 & -6 \\ 0 & 14 & 42 \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{cc|c} 3 & 2 & -6 \\ 0 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 3 & 0 & -12 \\ 0 & 1 & 3 \end{array}\right) \implies x^* = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

The least-squares error is then $\|Ax^* - b\| = 0$, so in fact x^* is a solution of $Ax = b$.

Ex 13.6 (Another least-squares problem)
Find all least-squares solution $x^\star$ of the system $Ax = b$ and their least square errors $\|Ax^\star - b\|$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

Solution:
We compute

$$A^T A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 14 \\ 4 \\ 10 \end{pmatrix}$$

and solve $A^T Ax = A^T b$ with row reduction:

$$\left(\begin{array}{ccc|c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & 1 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 0 & 2 & 10 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\implies x^* = \begin{pmatrix} -t+5 \\ t-3 \\ t \end{pmatrix}$$

These are all the least-squares solutions. To get the least-squares error, we can pick any value of t and compute that $\|Ax^* - b\| = \sqrt{20}$ (it must be the same for all t, otherwise these wouldn't all be least-squares solutions).

Ex 13.7 (QR decomposition for a least-square problem)
Consider

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}.$$

a) Show that

$$A = \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}.$$

b) Use this QR decomposition of A to find the least squares solution to the equation $Ax = b$.

Solution:

a) We write

$$Q = \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix} \qquad R = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}.$$

It can be easily checked that $A = QR$.

As the columns of Q are orthonormal and R is an upper triangular matrix, this is indeed a QR decomposition of the matrix A.

b) In order to make use of this decomposition for the least-squares problem, note that the equation system $A^TAb^* = A^Tb$ is equivalent to the system $(R^TQ^TQRb^* =)R^TRb^* = R^TQ^Tb$. Moreover, R^T is invertible, so the least squares solutions for the equation $Ax = b$ agree with the solutions of $Rx^* = Q^Tb$.

We hence have to solve the system

$$\begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} x^* = \begin{pmatrix} 7 \\ -1 \end{pmatrix},$$

giving

$$x^* = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

Ex 13.8 (Linear regression)

(a) Find the straight line that best approximates (in the sense of least squares) the following data points in $\mathbb{R}^2$: $(2,1),\quad (5,2),\quad (7,3),\quad (8,3)$

(b) Draw a picture that illustrates the data points and the line that best approximates them.

Solution:
The vertical distance between a point (x_0, y_0) and the line $y = ax + b$ equals $(ax_0 + b) - y_0$, so the question is to minimize

$$Q = (2a + b - 1)^2 + (5a + b - 2)^2 + (7a + b - 3)^2 + (8a + b - 3)^2.$$

As seen in the lecture, this is equivalent to the least-squares problem $Ax = b$ with

$$x = \begin{pmatrix} a \\ b \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$$

So we solve $A^T A x^* = A^T b$ as usual.

$$A^T A = \begin{pmatrix} 142 & 22 \\ 22 & 4 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 57 \\ 9 \end{pmatrix}$$

The row reduction is a bit annoying, so let's just use the formula for the inverse of a 2×2 matrix:

$$x^* = (A^T A)^{-1} A^T b = \frac{1}{84} \begin{pmatrix} 4 & -22 \\ -22 & 142 \end{pmatrix} \begin{pmatrix} 57 \\ 9 \end{pmatrix} = \frac{1}{84} \begin{pmatrix} 30 \\ 24 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

So the line that fits best is $y = \frac{5}{14}x + \frac{4}{14}$ or $5x - 14y = -4$. Hopefully you can see that in your picture :-)

Ex 13.9 (Linear regression)

Assume that you measure the measure the temperature near a chemical experiment at times $t = 1, 2, 3, 4, 5, 6$. The measurements y (ordered by time) that you obtain are $20, 30, 35, 40, 45, 45$. Find a affine function $f(t) = y$ approximating your data with minimal least square error. Also, give the value of the least square error.

Solution: Follow the same procedure as in Ex. 13.8 with

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \quad \text{and } b = \begin{pmatrix} 20 \\ 30 \\ 35 \\ 40 \\ 45 \\ 45 \end{pmatrix}$$

This yields the least square solution $x^* = \begin{pmatrix} 5 \\ 18 + \frac{1}{3} \end{pmatrix}$. Hence the linear function approximating the data with minimal least square error is

$$f(t) = 5t + 18 + \frac{1}{3}$$

and the least square error for f is:

$$\|Ax^* - b\| = \frac{10}{\sqrt{3}}.$$

Ex 13.10 (Repetition of old topics with application in Section 7)

(a) Prove that the set of symmetric matrices in $\mathbb{R}^{n\times n}$ are a subspaces of $\mathbb{R}^{n\times n}$.

(b) Prove that the dimension of this subspaces is $\frac{n(n+1)}{2}$

(c) What is the dimension of the space of anti-symmetric matrices?

Solution:

(a): The zero-matrix is symmetric. Sums of symmetric matrices are symmetric, and, a scalar times a symmetric matrix yields a symmetric matrix. Hence the symmetric matrices form a subspace of the space of all matrices in $\mathbb{R}^{n\times n}$.

(b): Let $e_{ij} \in \mathbb{R}^{n\times n}$ be the matrix with 1 at the ij-th and at the ji-th position and 0 everywhere else. And set

$$B := \{e_{11}, \dots, e_{nn}\} \cup \{e_{ij} + e_{ji} : i, j = 1, \dots, n, i < j\}.$$

It is clear that B is linearly independent and spans the space of symmetric matrices. Consequently, it forms a basis for the symmetric matrices. Then, by counting, we see that $|B| = \frac{n(n+1)}{2}$ as claimed.

(c): The dimension is $\frac{n(n+1)}{2} - n$.

Explanation: in part (b), the dimension was $\frac{n(n+1)}{2}$ because for a symmetric matrix, we can choose all diagonal entries freely as well as all entries under the diagonal. That is a total of $\frac{n(n+1)}{2}$ entries that we can choose freely. Then, because of symmetry, the entries above the diagonal are prescribed ($a_{ij} = a_{ji}$). This thought is what led to the choice of basis above and hence to the dimension of space of symmetric matrices.

Now for anti-symmetric matrices: Recall that anti-symmetric matrices have only zeros on the diagonal. So here, we can only freely choose the entries under the diagonal. That is n less entries than we were allowed to choose above. So the dimension is $\frac{n(n+1)}{2} - n$.

For a rigorous proof, just write down the basis for the space of anti-symmetric matrices that arises from this thought. (e.g. choose the basis entries to be all $\tilde{e}_{ij}$ for $i < j$, where $\tilde{e}_{ij}$ is the $n \times n$ matrix that has a 1 at position ij and a -1 at position ji.)

Ex 13.11 (Two quick proofs)

a) Let $A \in \mathbb{R}^{n\times n}$. Show that $A^T = A$ if and only if $Ax \cdot y = x \cdot Ay$ for all $x, y \in \mathbb{R}^n$.

b) Let $Q, U \in \mathbb{R}^{n\times n}$ be orthogonal matrices. Show that QU and Q^{-1} are also orthogonal.

Solution:

a) If $A^T = A$, then for all $x, y \in \mathbb{R}^n$ we have $Ax \cdot y = (Ax)^T y = x^T A^T y = x^T Ay = x \cdot Ay$. Conversely, if the above equality holds for all $x, y \in \mathbb{R}^n$, then in particular $Ae_i \cdot e_j = e_i \cdot Ae_j$ for all $i, j \in \{1, \dots, n\}$. Note that $Ae_i \cdot e_j = a_{ji}$ and $e_i \cdot Ae_j = a_{ij}$, so if those terms are equal, then $A^T = A$.

b) If Q, U are orthogonal, then $Q^{-1} = Q^T$ and $U^{-1} = U^T$, so that

$$(QU)^{-1} = U^{-1}Q^{-1} = U^T Q^T = (QU)^T.$$

Moreover, $(Q^{-1})^{-1} = (Q^T)^{-1} = (Q^{-1})^T$, so that also Q^{-1} is orthogonal.

Ex 13.12 (Multiple choice and True/False questions)

a) Let the matrix $A = \begin{pmatrix} -3 & -2 \\ 0 & 1 \\ 2 & -3 \end{pmatrix}$ and the vector $b = \begin{pmatrix} -6 \\ 11 \\ 17 \end{pmatrix}$.

Then the solution in the sense of the least squares $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ of the equation $Ax = b$ is such that

$(A)\quad x_2^* = -2 \qquad (B)\quad x_2^* = 3 \qquad (C)\quad x_2^* = -1 \qquad (D)\quad x_2^* = 1$

b) Decide whether the following statements are always true or if they can be false.

(i) Let $y \in \mathbb{R}^n$ and W be a subspace of $\mathbb{R}^n$. Then $y - \mathrm{proj}_W(y)$ is orthogonal to W.

(ii) If W is a subspace of $\mathbb{R}^n$, then $\mathrm{proj}_W \circ \mathrm{proj}_W = \mathrm{proj}_W$, where $\circ$ denotes the composition of maps.

(iii) If $A = QR$ and Q has orthonormal columns, then $R = Q^T A$.

(iv) A least-squares solution of $Ax = b$ is a vector $x_0 \in \mathbb{R}^n$ such that $Ax_0 = \mathrm{proj}_{\mathrm{Col}(A)}(b)$.

(v) If $b \in \mathrm{Col}(A)$, then the least-squares solutions are exactly the solution of the equation $Ax = b$.

(vi) The line of regression is unique provided we have measurements for at least two different inputs.

Solution:

a) **(A)** Solving $A^T A x = A^T b$ yields the answer.

b) Decide whether the following statements are always true or if they can be false.

(i) **True**. This follows from the orthogonal projection theorem.

(ii) **True**. This follows from the two facts that $\mathrm{proj}_W(x) \in W$ for all $x \in W$ and that $\mathrm{proj}_W(y) = y$ whenever $y \in W$.

(iii) **True**. We saw in the course that $Q^T Q = I_n$ when $Q \in \mathbb{R}^{m \times n}$. Thus we can multiply $A = QR$ by Q^T from the right.

(iv) **True**. We know that $\mathrm{proj}_{\mathrm{Col}(A)}(b)$ gives the closest point in $\mathrm{Col}(A)$ to b.

(v) **True**. If $Ax = b$ has a solution, then $\|Ax - b\|$ has the minimal value 0. Any least-squares solution thus satisfies $\|Ax - b\| = 0$, which is equivalent to $Ax = b$.

(vi) **True**. The matrix for the corresponding least-squares problem then has two linearly independepnt columns and therefore the least-squares solution is unique.