MATH-111(en) FALL 2024
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SOLUTIONS for Homework 12

Ex 12.1 (Diagonalizability)

a. Let A be a 3×3 matrix satisfying $A^3 = I_3$. Answer the following two questions: (i) What is the dimension of $\mathrm{Ker}(A)$? (ii) Is 0 an eigenvalue of A?

b. Let A be a 3×3 with characteristic polynomial equal to $\chi_A(\lambda) = (\lambda - 1)^2(\lambda + 1)$. Which of the following statements is true?

☐ A must be diagnoalizable.

☐ $\sigma(A) = \{-1, 1\}$

☐ A cannot be diagonalizable.

☐ In case A is diagonalizable, then there exist linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$ each satisfying $Av_i = -v_i$.

Solution:

a. Correct answer: $\dim \mathrm{Ker} A = 0$ and 0 is not an eigenvalue of A.

Reason: If 0 were an eigenvalue of A, then there exists $v \neq 0$ so that $Av = 0$. But then $A^3 v = 0$ and hence $I_3 v = 0$. This implies that $v = 0$. A contradiction. By the same argument $Av \neq 0$ for $v \neq 0$. Therefore, $\mathrm{Ker}(A) = \{0\}$ and hence $\dim \mathrm{Ker}(A) = 0$.

b. The only correct response $\sigma(A) = \{-1, 1\}$.

Reason: A having the characteristic polynomial $\chi_A(\lambda) = (\lambda - 1)^2(\lambda + 1)$ only tells us its eigenvalues and it does not tell us the dimension of the corresponding eigenspace to 1. So, as A is diagonizable if and only if the eigenspace corresponding to the eigenvalue 1 must have dimension 2, by simply finding examples for which this is *and* is not the case, we've shown that the first and third statements are false. An example of this holding is simply the diagonal matrix with diagonals $1, 1, -1$ while an example of where this does not hold is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Finally, the last statement holds only if the eigenspace corresponding to the eigenvalue -1 has dimension 2. However, this is not possible as the algebraic multiplicity of -1 is 1 and so, the dimension of the eigenspace is 1.

Ex 12.2 (Inner product calculations)

Let

$$u = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \qquad v = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}.$$

a) Calculate $u \cdot u$, $v \cdot v$, $u \cdot v$, $\|u\|$, and $\|v\|$.

b) Normalize u and v (i.e., find a unit vector with the same direction).

c) Find the distance between u and v, and find the cosine of the angle between them.

d) Find a basis of the space orthogonal to the plane spanned by u and v.

Solution:

a)

$$
\begin{gathered}
u \cdot u = 3 \cdot 3 + (-1) \cdot (-1) + 5 \cdot 5 = 35 \\
v \cdot v = 6 \cdot 6 + (-2) \cdot (-2) + 3 \cdot 3 = 49 \\
u \cdot v = 3 \cdot 6 + (-1) \cdot (-2) + 5 \cdot 3 = 35 \\
\|u\| = \sqrt{u \cdot u} = \sqrt{35}, \qquad \|v\| = \sqrt{v \cdot v} = \sqrt{49} = 7
\end{gathered}
$$

b)

$$
\frac{u}{\|u\|} = \frac{1}{\sqrt{35}} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \qquad \frac{v}{\|v\|} = \frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}
$$

c) The distance can be computed by

$$
\|u - v\| = \sqrt{(3-6)^2 + (-1-(-2))^2 + (5-3)^2)} = \sqrt{9+1+4} = \sqrt{14}
$$

We can compute the angle using $u \cdot v = \|u\| \|v\| \cos(\alpha(u, v))$ as follows:

$$
\cos(\alpha(u, v)) = \frac{u \cdot v}{\|u\| \|v\|} = \frac{35}{\sqrt{35}\sqrt{49}} = \frac{\sqrt{35}}{7} \left(= \sqrt{\frac{5}{7}} \right).
$$

d) Note that the space orthogonal to $\mathrm{Span}(u, v)$ is a line, i.e., one-dimensional, so any non-zero vector that is orthogonal to u and v will form a basis. Let

$$
w = \begin{pmatrix} a \\ b \\ c \end{pmatrix}
$$

be such a vector, meaning that $u \cdot w = 0$ and $v \cdot w = 0$. This yields a linear system

$$
\begin{aligned}
3a - b + 5c &= 0 \\
6a - 2b + 3c &= 0
\end{aligned}.
$$

A non-trivial solution of this system is given by

$$
w = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.
$$

Since we are in the 3-dimensional case, we could also use the cross product (not part of the course): From two vectors u and v this product yields a third vector $u \times v$ that is orthogonal to both u and v, and has length $\|u \times v\| = \|u\| \, \|v\| \, \sin(u, v)$. Hence in this case

$$
u \times v = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \\ 0 \end{pmatrix}
$$

gives another basis of the orthogonal space.

Ex 12.3 (An orthogonal basis)
Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}, \qquad u = \begin{pmatrix} 10 \\ 4 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

Show that $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^3$ and determine $[u]_\mathcal{B}$ and $[v]_\mathcal{B}$, i.e. represent them in the basis $\mathcal{B}$.

Solution:
To see that it is an orthogonal set, take the inner product of every pair and check that you get 0. As seen in the lecture, this already implies that $\mathcal{B}$ is an independent set, and 3 independent vectors in $\mathbb{R}^3$ are always a basis.
To represent a vector in an orthogonal basis, we don't have to do a row reduction like for other bases, because the orthogonality lets us use simple formulas:

$$u = \frac{u \cdot b_1}{b_1 \cdot b_1} b_1 + \frac{u \cdot b_2}{b_2 \cdot b_2} b_2 + \frac{u \cdot b_3}{b_3 \cdot b_3} b_3 = \frac{-1}{2} b_1 + \frac{17}{3} b_2 + \frac{-13}{6} b_3$$

$$\implies \quad [u]_\mathcal{B} = \begin{pmatrix} -1/2 \\ 17/3 \\ -13/6 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -3 \\ 34 \\ -13 \end{pmatrix}$$

$$v = \frac{v \cdot b_1}{b_1 \cdot b_1} b_1 + \frac{v \cdot b_2}{b_2 \cdot b_2} b_2 + \frac{v \cdot b_3}{b_3 \cdot b_3} b_3 = \frac{5}{2} b_1 + \frac{2}{3} b_2 + \frac{-1}{6} b_3$$

$$\implies \quad [v]_\mathcal{B} = \begin{pmatrix} 5/2 \\ 2/3 \\ -1/6 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 15 \\ 4 \\ -1 \end{pmatrix}$$

Ex 12.4 (Another orthogonal basis)
Consider the vectors

$$u = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \quad x = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

(a) Show that $\{u, v, w\}$ is an orthogonal basis of $\mathbb{R}^3$.

(b) Write the vector x as a linear combination of u, v and w.

Solution:

(a) We check that $u \cdot v = u \cdot w = v \cdot w = 0$, thus $\{u, v, w\}$ is an orthogonal basis of $\mathbb{R}^3$.

(b) We find

$$x = \frac{x \cdot u}{u \cdot u} u + \frac{x \cdot v}{v \cdot v} v + \frac{x \cdot w}{w \cdot w} w = \frac{4}{3} u + \frac{1}{3} v + \frac{1}{3} w.$$

Ex 12.5 (Properties of the orthogonal complement)
Let $W \subset \mathbb{R}^n$ be a subspace and $W^\perp$ be its orthogonal complement. Show the following statements:

- Lemma 6.2:
  (i) $W^\perp$ is a subspace of $\mathbb{R}^n$. Moreover, $W \cap W^\perp = \{0\}$.
  (ii) If $\mathcal{B}$ spans W, then $W^\perp = \{z \in \mathbb{R}^n : z \cdot b = 0 \quad \forall b \in \mathcal{B}\}$.
- Theorem 6.4: $\dim(W^\perp) = n - \dim(W)$.
  Hint: Let $b_1, ..., b_k$ be a basis of W and M the matrix whose columns are the b_i. Check that $W^\perp = \text{Ker}(M)$ and use Theorem 6.3.

Solution:

a) Let $w \in W$ be an arbitrary element. Then $0 \cdot w = 0$, so that $0 \in W^\perp$. If $x, y \in W^\perp$ and $\lambda \in \mathbb{R}$, then $x \cdot w = 0$ and $y \cdot w = 0$ and therefore $(\lambda x + y) \cdot w = \lambda x \cdot w + y \cdot w = \lambda 0 + 0 = 0$ and therefore $\lambda x + y \in W^\perp$. Hence $W^\perp$ is a subspace of $\mathbb{R}^n$. As the zero vector belongs to every subspace, it only remains to show that if $w \in W \cap W^\perp$, then $w = 0$. For such w we have $0 = w \cdot w = \|w\|^2$ and therefore $w = 0$.

b) If $\mathcal{B}$ spans W, then in particular $\mathcal{B} \subset W$ and therefore $W^\perp \subset \{z \in \mathbb{R}^n : z \cdot b = 0 \quad \forall b \in \mathcal{B}\}$ (the right-hand side set has less constraints). To prove the reverse inclusion, let $w \in W$ and write $w = \sum_{i=1}^k \lambda_i b_i$ for some $b_i \in \mathcal{B}$. If $z \cdot b = 0$ for all $b \in \mathcal{B}$, we get that

$$z \cdot w = \sum_{i=1}^{k} \lambda_i \underbrace{z \cdot b_i}_{=0} = 0$$

and therefore $z \in W^\perp$.

c) Let $\mathcal{B} = \{b_1, \ldots, b_k\}$ be a basis for W and write those vectors as columns in a matrix M. This matrix is of size $n \times k$ and has rank k. We know from b) and Theorem 6.3 that $W^\perp = \text{col}(M)^\perp = \text{Ker}(M^T)$. The matrix M^T is of size $k \times n$ and by the rank theorem we know that $n = \text{Rank}(M^T) + \dim(\text{Ker}(M^T)) = k + \dim(W^\perp)$. Since $k = \dim(W)$ this proves the claim.

Ex 12.6 (F^TF vs. FF^T for matrices with orthogonal columns)
Consider the matrix

$$F = \begin{pmatrix} 1 & 2 \\ -4 & 1/2 \end{pmatrix}.$$

Compute F^TF and FF^T. Are these two matrices equal ?

Solution:

$$F^TF = \begin{pmatrix} 17 & 0 \\ 0 & 17/4 \end{pmatrix}, \quad FF^T = \begin{pmatrix} 5 & -3 \\ -3 & 65/4 \end{pmatrix}.$$

These matrices are not equal.

Ex 12.7 (Orthogonality and projections) Prove the following statements about orthogonality and projections:

(i) Every orthogonal set that does not contain the zero vector is independent.
(This implies that in particular orthonormal sets are independent.)

(ii) The orthogonal projection from $\mathbb{R}^n$ onto a linear subspace $W \subset \mathbb{R}^n$ is a linear map.

Solution:
i) Let $S = \{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$ be an orthogonal set which doesn't contain 0. We will show S is linearly independent.
Suppose $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ is such that

$$\lambda_1 v_1 + \cdots + \lambda_k v_k = 0.$$

Then, for any $i = 1, \ldots, k$, we have

$$0 = v_i \cdot 0 = v_i \cdot (\lambda_1 v_1 + \cdots + \lambda_k v_k) = \lambda_1 v_i \cdot v_1 + \cdots + \lambda_k v_i \cdot v_k.$$

Since S is an orthogonal set, $v_i \cdot v_j = 0$ for all $i \neq j$ and thus, all but the i-th term in the above sum vanishes and we have

$$0 = \lambda_i v_i \cdot v_i = \lambda_i \|v_i\|^2.$$

However, as we had assumed $v_i \neq 0$, it follows that $\|v_i\|^2 > 0$ and so, $\lambda_i = 0$ and consequently S is linearly independent.
ii) Taking $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, it suffices to show

$$\mathrm{proj}_W(x_1 + \lambda x_2) = \mathrm{proj}_W(x_1) + \lambda \mathrm{proj}_W(x_2). \tag{1}$$

Recall that, for $x \in \mathbb{R}^n$, the orthogonal projection of x on to W: $\mathrm{proj}_W(x)$ is the unique element of W such that $x - \mathrm{proj}_W(x) \in W^\perp$. Thus, in order to show Eqaution (1), it suffices to show

(a) $\mathrm{proj}_W(x_1) + \lambda \mathrm{proj}_W(x_2) \in W$, and

(b) $(x_1 + \lambda x_2) - (\mathrm{proj}_W(x_1) + \lambda \mathrm{proj}_W(x_2)) \in W^\perp$.

(a) is clearly true since $\mathrm{proj}_W(x_1)$ and $\mathrm{proj}_W(x_2)$ are both in W so any linear combination of them will also be contained in W.
In order to show (b), we need to show that, for all $w \in W$,

$$((x_1 + \lambda x_2) - (\mathrm{proj}_W(x_1) + \lambda \mathrm{proj}_W(x_2))) \cdot w = 0.$$

Indeed,

$$\begin{aligned}
((x_1 + \lambda x_2) - (\mathrm{proj}_W(x_1) + \lambda \mathrm{proj}_W(x_2))) \cdot w &= ((x_1 - \mathrm{proj}_W(x_1)) + \lambda(x_2 - \mathrm{proj}_W(x_2))) \cdot w \\
&= (x_1 - \mathrm{proj}_W(x_1)) \cdot w + \lambda((x_2 - \mathrm{proj}_W(x_2)) \cdot w) \\
&= 0 + 0 = 0
\end{aligned}$$

where the last line follows as $x_1 - \mathrm{proj}_W(x_1)$ and $x_2 - \mathrm{proj}_W(x_2)$ are in $W^\perp$ by the definition of projection.

Ex 12.8 (Projection onto a subspace)
Let

$$u = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Determine the orthogonal projection $\mathrm{proj}_W(u)$ of u onto the subspace W spanned by v_1, v_2. Give it both in the basis $\mathcal{B} = \{v_1, v_2\}$ of W and in the standard basis of $\mathbb{R}^3$.

Solution:
Because v_1 and v_2 are orthogonal, we can simply use the formula

$$\mathrm{proj}_W(u) = \frac{u \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{u \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{1}{2} v_1 + \frac{6}{3} v_2$$

Denoting the standard basis by $\mathcal{E}$, this means that

$$[\mathrm{proj}_W(u)]_{\mathcal{B}} = \begin{pmatrix} 1/2 \\ 2 \end{pmatrix}, \qquad [\mathrm{proj}_W(u)]_{\mathcal{E}} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 2 \\ 3/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

Ex 12.9 (The row space and the kernel)
Consider an $m \times n$ matrix A.

a) Prove that every vector x in $\mathbb{R}^n$ can be written uniquely as $x = p + u$ where p belongs to $\mathrm{Row}(A)$ and u belongs to $\mathrm{Ker}(A)$.

b) Afterwards, show that if the equation $Ax = b$ is consistent, then there is a unique p in $\mathrm{Row}(A)$ such that $Ap = b$.
Hint: For uniqueness, use Lemma 6.2 (b).

Solution:

a) From Theorem 6.3 we deduce that $\mathrm{Row}(A)^\perp = \mathrm{Ker}(A)$, so that the claim follows from the orthogonal decomposition theorem (Theorem 6.7) applied to the subspace $\mathrm{Row}(A)$.

b) Assume that the system $Ax = b$ is consistent. Let x be a solution. Due to a) we can decompose it as $x = p + u$ with $p \in \mathrm{Row}(A)$ and $u \in \mathrm{Ker}(A)$. Then $Ap = A(x - u) = Ax - Au = b - 0 = b$. Thus the equation $Ax = b$ has at least a solution p in $\mathrm{Row}(A)$.

Let now p_1 and p_2 be two solutions to $Ax = b$ such that $p_1, p_2 \in \mathrm{Row}(A)$.

Then $p_2 - p_1$ belongs to $\mathrm{Ker}(A)$ since

$$A(p_2 - p_1) = Ap_2 - Ap_1 = b - b = 0.$$

Thus $p_2 - p_1$ is in $(\mathrm{Row}(A))^\perp \cap \mathrm{Row}(A)$. Applying Ex. 12.3 we find that $p_2 - p_1 = 0$, which shows uniqueness.

Ex 12.10 (Closest point in a column space)
Let A be the following matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

1. Show that the columns of A are an orthogonal set.

2. Write U, the matrix made of the normalized columns vectors of A.

3. Find the closest point to $y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathrm{Col}(U)$ and the distance from $b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ to $\mathrm{Col}(U)$.

Solution:

1. To show that the columns of A are an orthogonal set all we have to do is to check that $A^T A$ is a diagonal matrix. Here:

$$A^T A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

2. In order to find the matrix U one can notice that the diagonal of the matrix $A^T A$ reads the squared norm of the matrix A's columns. So we have:

$$U = \begin{pmatrix} 1/\sqrt{2} & -1/2 & 1/\sqrt{6} \\ 0 & 1/2 & 2/\sqrt{6} \\ -1/\sqrt{2} & -1/2 & 1/\sqrt{6} \\ 0 & 1/2 & 0 \end{pmatrix}.$$

3. The closest point to y in $\mathrm{Col}(U)$ is the projection (denoted as $\hat{y}$) of y on $\mathrm{Col}(U)$. The columns of U being orthonormal, Theorem 6.8 tells us that $\hat{y} = U\,U^T\,y$. That is:

$$\begin{aligned} \hat{y} &= \begin{pmatrix} 11/12 & 1/12 & -1/12 & -1/4 \\ 1/12 & 11/12 & 1/12 & 1/4 \\ -1/12 & 1/12 & 11/12 & -1/4 \\ -1/4 & 1/4 & -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2/3 \\ 4/3 \\ 2/3 \\ 0 \end{pmatrix}. \end{aligned}$$

The distance from b to $\mathrm{Col}(U)$ is $\|b - \hat{b}\|$, where $\hat{b}$ is the projection of b on $\mathrm{Col}(U)$,

$$\hat{b} = U\,U^T b = \begin{pmatrix} 1/2 \\ 5/2 \\ 1/2 \\ 1/2 \end{pmatrix},$$

thus $\|b - \hat{b}\| = \sqrt{3}$.

Ex 12.11 (Distance to different subspaces)

Let

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

Compute the distance from u to the line spanned by v_1, and the distance from u to the plane spanned by v_1 and v_2.

Solution:

Let $L = \mathrm{Span}(v_1)$ and $P = \mathrm{Span}(v_1, v_2)$. We calculate these distances using the fact that the orthogonal projection of a vector on a subspace is the point in that subspace closest to that vector. So

$$\mathrm{dist}(u, L) = \|u - \mathrm{proj}_L(u)\|, \qquad \mathrm{dist}(u, P) = \|u - \mathrm{proj}_P(u)\|$$

The projection onto the line L is given by

$$\operatorname{proj}_L(u) = \frac{u \cdot v_1}{v_1 \cdot v_1} v_1 = \frac{3}{5} v_1 = \frac{1}{5}\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

$$\implies \quad \operatorname{dist}(u, L) = \| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \| = \| \frac{1}{5}\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \| = \frac{1}{5}\sqrt{2^2 + (-1)^2 + 5^2} = \frac{\sqrt{30}}{5}$$

Moreover, since v_1 and v_2 are orthogonal, we know that

$$\operatorname{proj}_P(u) = \frac{u \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{u \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{3}{5} v_1 + \frac{1}{9} v_2 = \frac{1}{45}\begin{pmatrix} 17 \\ 59 \\ 10 \end{pmatrix}$$

$$\implies \quad \operatorname{dist}(u, P) = \| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{45}\begin{pmatrix} 17 \\ 59 \\ 10 \end{pmatrix} \| = \| \frac{1}{45}\begin{pmatrix} 28 \\ -14 \\ 35 \end{pmatrix} \| = \frac{1}{45}\sqrt{28^2 + 14^2 + 35^2} = \frac{7}{3\sqrt{5}}$$

Ex 12.12 (Multiple choice and True/False questions)
a) Let $A \in \mathbb{R}^{3\times 3}$. Which of the following sets of eigenvalues is possible?

(A) $\{1, 1+i, 2-i\}$, (B) $\{1, 2, 4i\}$, (C) $\{0, 3-i, 3+i\}$, (D) $\{i, 3-i, 3+i\}$.

b) Decide whether the following statements are always true or if they can be false.

(i) Let $u, v, w \in \mathbb{R}^n$. If $u \cdot v = 0$ and $v \cdot w = 0$, then $u \cdot w \neq 0$.

(ii) Let $u, v \in \mathbb{R}^n$. If the distance between u and v equals the distance between u and $-v$, then u and v are orthogonal.

(iii) If $A \in \mathbb{R}^{n\times n}$, then $\operatorname{Col}(A) = \operatorname{Ker}(A)^\perp$.

(iv) Let W be a subspace of $\mathbb{R}^n$. If x is orthogonal to every element of a basis for W, then $x \in W^\perp$.

(v) If $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then $\|\lambda x\| = \lambda \|x\|$.

(vi) The orthogonal projection of u onto v is the same as the orthogonal projection of u onto av for any $a \neq 0$.

(vii) If W is a subspace of $\mathbb{R}^n$ and $u \in W$, then $\operatorname{proj}_W(u) = u$.

(viii) Let A be an $n \times n$ matrix. The columns of A form an orthonormal basis of $\mathbb{R}^n$ if and only if $\det(A) = 1$.

(ix) If $A^T A = I$, then A must be square.

(x) A square matrix has orthonormal columns if and only if it has orthonormal rows.

(xi) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

(xii) A matrix with orthonormal columns is an orthogonal matrix.

(xiii) For each $y \in \mathbb{R}^n$ and each subspace W of $\mathbb{R}^n$, the vector $y - \operatorname{proj}_W y$ is orthogonal to W.

(xiv) If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T y$ is the orthogonal projection of y onto the column space of U.

Solution:
a) The answer is (C). Indeed, we know that for a matrix with real coefficients complex eigenvalues appear in pairs in the sense that also the conjugate is an eigenvalue (this rules out (A), (B) and (D)).

b)

(i) **False:** Consider for instance $u = e_1$, $v = e_2$, $w = e_3$ in $\mathbb{R}^3$.

(ii) **True:** If we have $\|u - v\| = \|u - (-v)\|$, then $(u - v) \cdot (u - v) = (u + v) \cdot (u + v)$, i.e.

$$u \cdot u - 2(u \cdot v) + v \cdot v = u \cdot u + 2(u \cdot v) + v \cdot v.$$

Hence $-2(u \cdot v) = 2(u \cdot v)$, which implies that $u \cdot v = 0$.
Remark: For questions relating the distance to the inner product, it is always a good idea to consider the squared distance to avoid the square-roots.

(iii) **False:** For instance, for $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ we have $\operatorname{Col} A = \operatorname{Span}(e_2) = \operatorname{Ker} A$ and $(\operatorname{Ker} A)^\perp = \operatorname{Span}(e_1)$.

(iv) **True:** This is a special case of Lemma 6.2 (b).

(v) **False:** The correct formula is $\|\lambda x\| = |\lambda|\|x\|$, so a counterexample is given by $x = e_1$ and $\lambda = -1$.

(vi) **True:** As seen in the course the a cancels out of the formula:

$$\operatorname{proj}_{av}(u) = \frac{u \cdot (av)}{(av) \cdot (av)}(av) = a\frac{a(u \cdot v)}{(a^2(v \cdot v))}v = \frac{u \cdot v}{v \cdot v}v = \operatorname{proj}_v(u).$$

(vii) **True:** By Theorem 6.7, there is a unique decomposition $u = p + o$ with $p = \operatorname{proj}_W(u) \in W$ and $o \in W^\perp$. But if $u \in W$, then $u = u + 0$ is such a decomposition, hence $\operatorname{proj}_W(u) = u$.

(viii) **False:** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has orthonormal columns but determinant -1, and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has determinant 1 but does not have orthogonal columns. So the implication doesn't work in either direction.

(ix) **False:** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(x) **True:** By Theorem 6.6, a matrix A has orthonormal columns if and only if $A^T A = I$. Since A is assumed to be square, this implies that A is invertible with $A^{-1} = A^T$. Hence also $AA^T = I$, or in other words $(A^T)^T A^T = I$. By the same theorem, this means that A^T has orthonormal columns, which means that A has orthonormal rows.

(xi) **False:** As $(\lambda v) \cdot (\mu w) = (\lambda\mu)(v \cdot w)$, scaling vectors doesn't affect the orthogonality relations between them.

(xii) **False:** The matrix also needs to be a square matrix.

(xiii) **True:** This is one of the crucial properties of the orthogonal projection (cf. Theorem 6.7).

(xiv) **True:** This is stated in Theorem 6.8.