MATH-111(en) FALL 2024
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SOLUTIONS for Homework 11

Ex 11.1 (Diagonalizable or not?)
Which of the following matrices are diagonalizable?

$$M_1 = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 6 & 0 \\ 1 & -2 & 2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution:
In each case, we have to determine the characteristic polynomial and its roots. If the roots are all distinct, and there are as many as the size of the matrix, then the matrix is diagonalizable. If they are not all distinct, then the matrix may or may not be diagonalizable. To find out we'll need to determine the dimensions of the eigenspaces, and see if they equal the multiplicity of the corresponding eigenvalue as a root of the characteristic polynomial.

$$\det(\lambda I - M_1) = \begin{vmatrix} \lambda - 2 & 0 \\ 1 & \lambda - 1 \end{vmatrix} = (\lambda - 2)(\lambda - 1)$$

Since this has two distinct roots, M_1 is diagonalizable.

$$\det(\lambda I - M_2) = \begin{vmatrix} \lambda - 1 & 2 & 0 \\ 0 & \lambda - 6 & 0 \\ -1 & 2 & \lambda - 2 \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda - 6 & 0 \\ 2 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda - 6)(\lambda - 2)$$

We find three distinct roots, so M_2 is diagonalizable.

$$\det(\lambda I - M_3) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)$$

The roots are not distinct, so we will have to find the eigenspace of $\lambda = 1$ and see if its dimension equals the multiplicity of $\lambda = 1$ as a root, which is 2. To find a basis of the eigenspace, we'll need a basis of $\mathrm{Ker}(M_1 - 1 \cdot I)$, which is easy because

$$M_3 - I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

so all solutions of $(M_3 - I)\mathbf{x} = 0$ are of the form $\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$, so a basis for $\mathrm{Ker}(M_1 - 1 \cdot I)$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
Hence the eigenspace of $\lambda = 1$ has dimension 1, which is less than its multiplicity as a root of

$(1-\lambda)^2(2-\lambda)$. Therefore M_3 is not diagonalizable.

Ex 11.2 (Diagonalization of a matrix)
Diagonalize the following matrix.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

If not stated otherwise, this means finding a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. In particular, you do not need to compute P^{-1}.

Solution:
We follow the method presented in the lecture:

- **Find the eigenvalues.** Laplace expansion with respect to the first row yields

$$\det(\lambda I - A) = \begin{pmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-2 & -1 \\ 1 & 0 & \lambda-1 \end{pmatrix} = (\lambda-2)^2(\lambda-1) \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2$$

- **Find bases of the eigenspaces.**

$$A - \lambda_1 \cdot I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{x} = \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$A - \lambda_2 \cdot I = A - \lambda_3 I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} -t \\ s \\ t \end{pmatrix} = s \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

- **Construct the similarity matrix P and diagonal matrix D.**

$$P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

- **Check $AP = PD$ (this step is optional).**

$$AP = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$PD = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Ex 11.3 (Déjà vu?)
Diagonalize the following matrix.

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution:

- **Find the eigenvalues.** Laplace expansion (first row, then second row) yields

$$\det(\lambda I - A) = (\lambda+2)^2(\lambda-2)^2 \;\Rightarrow\; \lambda_1 = -2, \lambda_2 = -2, \lambda_3 = 2, \lambda_4 = 2$$

- **Find bases of the eigenspaces.**

$$A - \lambda_1 \cdot I = A - \lambda_2 \cdot I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 2s-2t \\ s \\ t \\ 0 \end{pmatrix} = s \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \Rightarrow \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A - \lambda_3 \cdot I = A - \lambda_4 I = \begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ s \\ t \end{pmatrix} \Rightarrow \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- **Construct the similarity matrix P and diagonal matrix D.**

$$P = [\mathbf{v}_1 \; \mathbf{v}_2 \; \mathbf{v}_3 \; \mathbf{v}_4] = \begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

- **Check $AP = PD$. (this step is optional)**

$$AP = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 4 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$PD = \begin{pmatrix} 2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 4 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Ex 11.4 (More diagonalizability examples)
Consider

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \; C = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}, D = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \text{ and } E = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}.$$

1. For each matrix find out the eigenvalues and the corresponding eigenvectors.
   Hint: For C and D the rational root theorem (see Ex. 10.8) helps to find the eigenvalues.

2. Find out which ones are diagonalizable.

Solution:

- The solutions of the characteristic equation

  $$\det(\lambda I - A) = \lambda^2 - 8\lambda + 16 = 0$$

  are $\lambda_1 = \lambda_2 = 4$. So A has a unique eigenvalue ($\lambda = 4$) of algebraic multiplicity equal to 2. As the corresponding eigenspace $\mathrm{Ker}(A - 4I)$ is spanned by the vector

  $$\begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

  it has dimension 1. Thus, from the theorem of the lecture, A is not diagonalizable.

- As B is lower triangular, we can read off that its characteristic polynomial is $(\lambda-4)^2(\lambda-5)$. Hence its eigenvalues are 4 and 5. The eigenvalue 4 has algebraic multiplicity 2 and a one-dimensional eigenspace spanned by:

  $$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

  Again by the theorem of the lecture this implies that B is not diagonalizable. The eigenspace associated to the eigenvalue 5 is spanned by:

  $$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- C has three distinct eigenvalues: 1, 2 and 3. This shows that C is diagonalizable. The eigenspaces are all 1-dimensional and spanned by:

  $$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for } \lambda = 1, \quad \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix} \text{ for } \lambda = 2, \quad \begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix} \text{ for } \lambda = 3.$$

- The eigenvalues of D are 8 and 2, where 8 has multiplicity 1 and 2 has multiplicity 2. The eigenspace for 8 is spanned by

  $$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

  The eigenspace for 2 is 2-dimensional and spanned by

  $$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

  Thus, D is diagonalizable.

- E has one eigenvalue 5, with algebraic multiplicity 2. The eigenspace associated to 5 is 1-dimensional and spanned by
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
Thus E is not diagonalizable.

Ex 11.5 (Powers of a diagonalizable matrix)
Let $A = PDP^{-1}$ with $P \in \mathbb{R}^{n\times n}$ invertible and $D \in \mathbb{R}^{n\times n}$ a diagonal matrix. Show that for any $k \in \mathbb{N}$ it holds that $A^k = PD^kP^{-1}$.
Remark: Powers of a diagonal matrix are easy to calculate. We have seen in the course that we just need to take the corresponding powers of the diagonal elements.

Solution:
We prove the statement by induction on k. For $k = 1$ the statement is obvious. Next, for $k > 1$ we have

$$A^k = A^{k-1}A = (PD^{k-1}P^{-1})(PDP^{-1}) = PD^{k-1}(P^{-1}P)DP^{-1} = PD^{k-1}DP^{-1} = PD^kP^{-1}.$$

Ex 11.6 (Matrix representation of linear maps)
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 4z \\ 3x + 5y - 2z \\ x + y + 4z \end{pmatrix}.$$

Consider the ordered basis of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Find the matrix $M = \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$ that represents T in the basis $\mathcal{B}.[T(\mathbf{v})]_{\mathcal{B}} = M \cdot [\mathbf{v}]_{\mathcal{B}}$.

Solution:
Method 1: The columns of $\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]}$ are the vectors $[T(\mathbf{b}_1)]_{\mathcal{B}}$, $[T(\mathbf{b}_2)]_{\mathcal{B}}$, $[T(\mathbf{b}_3)]_{\mathcal{B}}$, and the vector $[T(\mathbf{b}_i)]_{\mathcal{B}}$ is the solution of $B\mathbf{x} = T(\mathbf{b}_i)$, where B is the matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. So we can use row reduction to find these columns. We can even find them simultaneously using the following row reduction:

$$\left(B \mid T(b_1)\, T(b_2)\, T(b_3) \right) \longrightarrow \left(I \mid \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} \right)$$

We have

$$\begin{pmatrix} T(b_1) & T(b_2) & T(b_3) \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 6 & 8 & 3 \\ 6 & 2 & 1 \end{pmatrix}.$$

So we do

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 4 & 0 & 0 \\ 1 & 1 & 0 & 6 & 8 & 3 \\ 1 & 0 & 0 & 6 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 4 & 0 & 0 \\ 0 & 0 & -1 & 2 & 8 & 3 \\ 0 & -1 & -1 & 2 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 4 & 0 & 0 \\ 0 & 1 & 1 & -2 & -2 & -1 \\ 0 & 0 & 1 & -2 & -8 & -3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 6 & 8 & 3 \\ 0 & 1 & 0 & 0 & 6 & 2 \\ 0 & 0 & 1 & -2 & -8 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 2 & 1 \\ 0 & 1 & 0 & 0 & 6 & 2 \\ 0 & 0 & 1 & -2 & -8 & -3 \end{array}\right)$$

$$\Rightarrow \underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{pmatrix} 6 & 2 & 1 \\ 0 & 6 & 2 \\ -2 & -8 & -3 \end{pmatrix}$$

Method 2: We saw in the lecture that if B is the matrix whose columns are the vectors of $\mathcal{B}$, and A is the matrix representing T in the standard basis, then

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = B^{-1}AB.$$

So we first find B^{-1} with row reduction:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

In the standard basis of $\mathbb{R}^3$, the matrix of T is

$$A = \begin{pmatrix} 0 & 0 & 4 \\ 3 & 5 & -2 \\ 1 & 1 & 4 \end{pmatrix},$$

so

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = B^{-1}AB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 4 \\ 3 & 5 & -2 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 1 \\ 0 & 6 & 2 \\ -2 & -8 & -3 \end{pmatrix}.$$

Ex 11.7 (Another matrix representation)
Let the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined by:

$$T(\mathbf{p}) = \begin{pmatrix} p(0) \\ p(0) \\ p(2) \end{pmatrix} \quad \text{for any polynomial } \mathbf{p} \in \mathbb{P}_2.$$

a) Find the matrix A of the linear transformation T in terms of the standard bases of $\mathbb{P}_2$ and $\mathbb{R}^3$.

b) Using the matrix A, determine the kernel and image of T.

Solution:

a) A Polynomial $\mathbf{p} \in \mathbb{P}_2$ is written as $\quad p(t) = a_0 + a_1 t + a_2 t^2 \quad$ and we have that

$$T(\mathbf{p}) = \begin{pmatrix} a_0 \\ a_0 \\ a_0 + 2a_1 + 4a_2 \end{pmatrix}$$

Given $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ the canonical basis of $\mathbb{P}_2$ with $p_1(t) = 1, \quad p_2(t) = t \quad$ and $\quad p_3(t) = t^2$, then the canonical matrix associated with T is

$$A = \begin{pmatrix} T(\mathbf{p}_1) & T(\mathbf{p}_2) & T(\mathbf{p}_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix} \quad .$$

b) To obtain the T kernel we determine $\operatorname{Ker} A$ by writing the solution set of $A\mathbf{x} = \mathbf{0}$ in parametric vector form. Since $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $\left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$ is a basis for $\operatorname{Ker} A$ and it follows that

$$\operatorname{Ker} T = \{\mathbf{p} \in \mathbb{P}_2 : \quad p(t) = \alpha \cdot (-2t + t^2) \quad \text{with} \quad \alpha \in \mathbb{R}\} \quad .$$

To get the image of T we define $\operatorname{Col} A$ by selecting the pivot columns of A.
Since $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$ is a basis of $\operatorname{Col} A$, we have

$$\operatorname{Ran} T = \left\{ \mathbf{x} \in \mathbb{R}^3 : \quad \mathbf{x} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \quad \text{with} \quad \alpha \in \mathbb{R} \quad \text{and} \quad \beta \in \mathbb{R} \right\} \quad .$$

Ex 11.8 (Partial proof of Theorem 5.10)
Let $A \in \mathbb{R}^{n \times n}$ and $\sigma(A) = \{\lambda_1, ..., \lambda_k\}$ (distinct list of eigenvalues). Prove that:

(1) If A is diagonalizable, then for each i:
$\operatorname{multgeom}_A(\lambda_i) = \operatorname{multalg}_A(\lambda_i)$.

(2) A is diagonalziable if and only if $\sum_{i=1}^{n} \operatorname{multgeom}_A(\lambda_i) = n$

Solution:

(1): Recall that for a diagonal $n \times n$ matrix D with diagonal entries d_i, the characteristic polynomial is the product $(\lambda - d_1)(\lambda - d_2)...(\lambda - d_n)$. Hence the eigenvalues are the d_i and their algebraic multiplicity is the number of times it shows up on the diagonal, that is, $\operatorname{multalg}_D(d_i) = \#\{j : d_i = d_j\}$.
On the other hand, e_i is an eigenvector of D for d_i. Therefore:
$\operatorname{multgeom}_D(d_i) = \#$ independent eigenvectors for $d_i = \#\{j : d_i = d_j\}$.
So $\operatorname{multalg}_D(d_i) = \operatorname{multgeom}_D(d_i)$.

Now if A is diagonalizable, then A is similar to a diagonal matrix D, and by Proposition 5.6: the eigenvalues of A and D as well as their algebraic and geometric multiplicities are the same.

(2) $\Rightarrow$: Before we even start the proof, let us observe the following: while for the first part (above) it was not relevant that the list of eigenvalues $\lambda_1, ..., \lambda_k$ is distinct, it does play a role in this part. Observe that the sum of geometric multiplicies would become larger if we summed the multiplicity of double occuring eigenvalues twice.
Assume that A is diagonalizable. By Thm.5.7, this means that A has an eigenbasis $\mathcal{B}$. This implies that the elements $b_1, ..., b_n$ of $\mathcal{B}$ are eigenvectors of A and that they are independent.

Each b_i must live in the eigenspace $E_A(\lambda_j)$ of some eigenvalue λ_j of A. (Several b_i might be eigenvectors of the same λ_j and hence live in the same eigenspace $E_A(\lambda_j)$.)
By independence of the b_i we get:
$\text{multgeom}_A(\lambda_j) = \dim E_A(\lambda_j) \geq \#\text{indep. eigenvectors in } E_A(\lambda_j) \geq \#\{b_i : b_i \in E_A(\lambda_j)\}$.
So $\sum_{i=1}^n \text{multgeom}_A(\lambda_i) = \sum_{i=1}^n \dim(E_A(\lambda_i)) \geq n$.
But also, since eigenvectors of different eigenvalues are always independent (Thm.5.4), the sum $\sum_{i=1}^n \dim(E_A(\lambda_i))$ must be $\leq n$. (Otherwise we would have more than n independent vectors in $\mathbb{R}^n$ which is not possible.)
Hence $\sum_{i=1}^n \text{multgeom}_A(\lambda_i) = \sum_{i=1}^n \dim(E_A(\lambda_i) = n$.

$\Leftarrow$: Assume that $\sum_{i=1}^n \text{multgeom}_A(\lambda_i) = n$. By definition of geometric mulitiplicity, the number of independent eigenvectors for the eigenvalue λ_i is $\text{multgeom}_A(\lambda_i)$. Moreover, by Thm.5.4, eigenvectors of distinct eigenvalues are always independent. Hence, we can find $\sum \text{multgeom}_A(\lambda_i)$ independent eigenvectors for A. But by assumption, this means that A has n independent eigenvectors. Since n indepenedent vectors in $\mathbb{R}^n$ always are a basis of $\mathbb{R}^n$, this means that A has an eigenbasis. Hence A is diagonalizable (Thm.5.7).

Ex 11.9 (Multiple choice and True/False questions)

a) i) Let A be a 3×3 matrix that has the eigenvalues -1, 1 and 2. Then

I) The rank of A is equal to

(A) 1 (B) 2 (C) 0 (D) 3

II) The determinant of $A^T A$ is equal to

(A) 2 (B) 4 (C) 0 (D) 3

III) The determinant of $A + I$ is equal to

(A) 1 (B) 6 (C) 0 (D) -1

IV) The determinant of A^{-1} is equal to

(A) -2 (B) -1 (C) 1 (D) $-1/2$

ii) Consider the matrices

$$A = \begin{pmatrix} 2 & 0 & 8 \\ 1 & 4 & -4 \\ 1 & 2 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 3 & -3 & 1 \end{pmatrix}.$$

Then the following matrices are diagonalizable:

(A) both A and B (B) only B (C) only A (D) neither A nor B.

iii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6x_1 + 2x_2 + 4x_3 \\ -3x_2 + x_3 \\ 2x_1 + 8x_2 - x_3 \end{pmatrix}.$$

Let $\mathcal{E}$ and $\mathcal{F}$ be two bases of $\mathbb{R}^3$ given by

$$\mathcal{E} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \qquad \text{and} \qquad \mathcal{F} = \left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\2\end{pmatrix} \right\}.$$

Then the matrix of T in the bases $\mathcal{E}$ (outgoing) and $\mathcal{F}$ (incoming) is

$$(A)\ \begin{pmatrix}3&2&2\\1&2&5\\2&1&0\end{pmatrix} \quad (B)\ \begin{pmatrix}3&1&1\\2&3&1\\2&5&0\end{pmatrix} \quad (C)\ \begin{pmatrix}3&1&2\\2&2&1\\2&5&0\end{pmatrix} \quad (D)\ \begin{pmatrix}3&1&2\\2&0&1\\1&2&0\end{pmatrix}.$$

b) In the following, let A be an $n \times n$ matrix. Decide whether the following statements are always true or if they can be false.

i) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors of A, then they correspond to distinct eigenvalues.

ii) If A is invertible, then it is diagonalizable.

iii) If A is not invertible, then it is not diagonalizable.

iv) If A has fewer than n distinct eigenvalues, then A is not diagonalizable.

v) If $AP = PD$ for some diagonal matrix D, then all columns of P are eigenvectors of A.

vi) If $AP = PD$ for some diagonal matrix D, then A is diagonalizable.

vii) A has diagonalizable if A has n eigenvectors.

viii) If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.

Solution:

a) i) The key to the answers is to note that A is diagonalizable and $A = PDP^{-1}$. Thus $\det(A) = \det(D) = -2$.

I) **The answer is (D).** Indeed, A is invertible.

II) **The answer is (B).** Indeed, $\det(A^T A) = \det(A)^2$.

III) **The answer is (C).** We know that -1 is an eigenvalue, so $A + I$ cannot be invertible.

IV) **The answer is (D).** Indeed, $\det(A^{-1}) = 1/\det(A)$.

ii) **The answer is (A).**

iii) **The answer is (C).**

b) i) **False:** We have seen two-dimensional eigenspaces, which contain two linearly independent eigenvectors that correspond to the same eigenvalue.

ii) **False:** $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ is invertible, but not diagonalizable.

iii) **False:** $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ is not invertible, but diagonalizable (because it is already diagonal).

iv) **False:** If $n > 1$, the identity matrix I_n has fewer than n distinct eigenvalues (namely one), but it is diagonalizable.

v) **False:** The equality holds for all choices of A and D when P is the zero matrix. But the columns of this P can never be eigenvectors by definition. (because they are all zero vectors).

vi) **False:** If we choose $P = 0$ again, the equality holds for an arbitrary diagonal matrix D and an arbitrary non-diagonalizable matrix A.

vii) **False:** To conclude that A is diagonalizable, the n eigenvectors must also be linearly independent.

viii) **True.** Indeed, let $v_j \in \mathbb{R}^n$ be the jth column of P and assume $v_j \neq 0$. Then $Av_j = APe_j = PDe_j = Pd_{jj}e_j = d_{jj}Pe_j = d_{jj}v_j$, where d_{jj} is the jth diagonal entry of D. Since $v_j \neq 0$, it is an eigenvector of A.