MATH-111(en) FALL 2024
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SOLUTIONS for Homework 10

Ex 10.1 (Calculus with complex numbers)

a) Find the real and imaginary part of $z = \dfrac{1}{1+i} - \dfrac{2+3i}{2+i}$.

b) Calculate the modulus of the following complex numbers:

$$z_1 = i, \quad z_2 = 2-2i, \quad z_3 = -4i.$$

c) Given $z \in \mathbb{C}$ we define $w \in \mathbb{C}$ by

$$w = \varepsilon \cdot \sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} + i \cdot \sqrt{\frac{|z| - \mathrm{Re}(z)}{2}},$$

where $\varepsilon = \pm 1$ is the sign of $\mathrm{Im}(z)$ (convention: the sign of zero equals 1). Show that $w^2 = z$, i.e., w is a square-root of z. Use this result to find a square-root of i.

Solution:

a) $\dfrac{1}{1+i} - \dfrac{2+3i}{2+i} = \dfrac{1-i}{(1+i)(1-i)} - \dfrac{(2+3i)(2-i)}{(2+i)(2-i)} = \dfrac{1-i}{2} - \dfrac{7+4i}{5}$
$= \dfrac{1}{2} - \dfrac{1}{2}i - \dfrac{7}{5} - \dfrac{4}{5}i = -\dfrac{9}{10} - \dfrac{13}{10}i.$

Hence $\mathrm{Re}(z) = -\frac{9}{10}$ and $\mathrm{Im}(z) = -\frac{13}{10}$.

b) We have $|z_1| = \sqrt{1} = 1$, $|z_2| = \sqrt{2^2 + (-2)^2} = \sqrt{8}$ and $|z_3| = \sqrt{(-4)^2} = 4$.

c) Using the formula for the complex product the real part of w^2 is given by

$$\begin{aligned}
\mathrm{Re}(w^2) &= \left(\varepsilon\sqrt{\frac{|z| + \mathrm{Re}(z)}{2}}\right)^2 - \left(\sqrt{\frac{|z| - \mathrm{Re}(z)}{2}}\right)^2 \\
&= \left(\frac{|z| + \mathrm{Re}(z)}{2}\right) - \left(\frac{|z| - \mathrm{Re}(z)}{2}\right) \\
&= \mathrm{Re}(z),
\end{aligned}$$

where we used that $\varepsilon^2 = 1$. The imaginary part of w^2 is given by

$$\begin{aligned}
\mathrm{Im}(w^2) &= 2\varepsilon \cdot \left(\sqrt{\frac{|z| + \mathrm{Re}(z)}{2}}\right) \cdot \left(\sqrt{\frac{|z| - \mathrm{Re}(z)}{2}}\right) \\
&= 2\varepsilon \cdot \sqrt{\left(\frac{|z| + \mathrm{Re}(z)}{2}\right) \cdot \left(\frac{|z| - \mathrm{Re}(z)}{2}\right)} \\
&= 2\varepsilon \cdot \sqrt{\frac{|z|^2 - \mathrm{Re}(z)^2}{4}} \\
&= \varepsilon \cdot |\mathrm{Im}(z)| = \mathrm{Im}(z)
\end{aligned}$$

since $\sqrt{|z|^2 - \mathrm{Re}(z)^2} = \sqrt{\mathrm{Im}(z)^2} = |\mathrm{Im}(z)|$ and $\varepsilon = \mathrm{sgn}(\mathrm{Im}(z))$. Note that in the above calculation all roots involved non-negative arguments since $|z| \geq |\mathrm{Re}(z)|$.

By definition we have $|i| = 1$, $\mathrm{Re}(i) = 0$ and $\mathrm{Im}(i) = 1$, so that $\sqrt{i} = \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}i$.

Ex 10.2 (The matrix of a linear map relative to two bases)
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 4x_1 + 3x_2 \\ 10x_1 - 8x_2 \\ x_1 + 2x_2 \end{pmatrix}.$$

Consider the bases $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$ and $\mathcal{C} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$. Give the matrix $M = \underset{\mathcal{C} \leftarrow \mathcal{B}}{[T]}$ that represents T going from $\mathcal{B}$ to $\mathcal{C}$; in other words, the matrix M such that $[T(\mathbf{v})]_{\mathcal{C}} = M \cdot [\mathbf{v}]_{\mathcal{B}}$.

Solution:
We first calculate the images of the basis vectors in $\mathcal{B}$ under T, finding that

$$T(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}, \qquad T(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 3 \\ -8 \\ 2 \end{pmatrix}.$$

Next we need to find the $\mathcal{C}$-coordinates of these vectors by solving two linear systems. As seen in the course, this can be done simultaneously by considering the augmented matrix

$$\left(\begin{array}{ccc|cc} -1 & -1 & 2 & 7 & 3 \\ 0 & 2 & 0 & 2 & -8 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -2 & 5 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 3 & 2 \end{array}\right).$$

Therefore

$$M = \begin{pmatrix} -2 & 5 \\ 1 & -4 \\ 3 & 2 \end{pmatrix}.$$

Ex 10.3 (Finding a basis for an eigenspace)
Find a basis for the eigenspace of the eigenvalue $\lambda = 3$ of the matrix

$$B = \begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}.$$

Solution:
We have to find a basis for $\ker(B - 3I)$, so we row reduce this matrix:

$$\begin{pmatrix} 4-3 & 2 & 3 \\ -1 & 1-3 & -3 \\ 2 & 4 & 9-3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s-3t \\ s \\ t \end{pmatrix} = s \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

This is the parametric vector form of the elements lying in the kernel, so we know that $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis.

Ex 10.4 (Some eigenvalues and eigenvectors)
Consider the matrices

$$C = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix}.$$

Find the eigenvalues of both, and give an eigenvector for each eigenvalue.

Solution:
We first calculate the characteristic polynomial[1]

$$\det(\lambda I - C) = \det \begin{pmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{pmatrix} = (\lambda-3)(\lambda-3) - (-1)\cdot(-1)$$

$$= \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4),$$

so $\lambda_1 = 2$ and $\lambda_2 = 4$ are eigenvalues of C. If you don't see the factorization of the last step, the roots of the characteristic polynomial can also be found with the well-known a-b-c formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
Then we row reduce $B - 2I$ and $B - 4I$ to get eigenvectors:

$$\begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3-4 & 1 \\ 1 & 3-4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then we do the same for D:

$$\det(\lambda I - D) = \det \begin{pmatrix} \lambda - 5 & -4 \\ 1 & \lambda - 1 \end{pmatrix} = (\lambda-5)(\lambda-1) - (-4)\cdot 1 = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2,$$

so $\lambda = 3$ is the only eigenvalue. Then:

$$\begin{pmatrix} 5-3 & 4 \\ -1 & 1-3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Ex 10.5 (More eigenvalues and eigenvectors)
Find the eigenvalues of

$$E = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

[1] In the course the characteristic polynomial was defined as $\chi_A(\lambda) = \det(\lambda I - A)$. We always have $\det(A - \lambda I) = (-1)^n \det(\lambda I - A)$, so these two polynomials have the same roots. For computations it might be faster to take $A - \lambda I$ to avoid the sign change for all entries of A, but for theoretical arguments the other formula is more useful as the leading coefficient is always equal to 1. To find eigenvalues, you can choose the formula you prefer.

and give an eigenvector for each.

Solution:
We first compute the characteristic polynomial:

$$\det(\lambda I - E) = \begin{vmatrix} \lambda+1 & 0 & -1 \\ 3 & \lambda-4 & 0 \\ 0 & 0 & \lambda-2 \end{vmatrix} = (1+\lambda)\begin{vmatrix} \lambda-4 & 0 \\ 0 & \lambda-2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & \lambda-4 \\ 0 & 0 \end{vmatrix}$$

$$= (\lambda+1)(\lambda-4)(\lambda-2) \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = 4, \quad \lambda_3 = 2;$$

$$E - \lambda_1 I = \begin{pmatrix} -1-(-1) & 0 & 1 \\ -3 & 4-(-1) & 0 \\ 0 & 0 & 2-(-1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -3 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} -3 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{3}t \\ t \\ 0 \end{pmatrix} = t \cdot \begin{pmatrix} \frac{5}{3} \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix};$$

$$E - \lambda_2 I = \begin{pmatrix} -1-4 & 0 & 1 \\ -3 & 4-4 & 0 \\ 0 & 0 & 2-4 \end{pmatrix} = \begin{pmatrix} -5 & 0 & 1 \\ -3 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -5 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};$$

$$E - \lambda_3 I = \begin{pmatrix} -1-2 & 0 & 1 \\ -3 & 4-2 & 0 \\ 0 & 0 & 2-2 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 1 \\ -3 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -3 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3}t \\ \frac{1}{2}t \\ t \end{pmatrix} = t \cdot \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}.$$

(in the first and last case one can also take the vector with fractional entries; we just multiplied it by 3 respectively 6 to have integer coordinates)

Ex 10.6 (Eigenvalues of AB and BA)
Let A and B be two $n \times n$ matrices. Show that AB and BA have the same eigenvalues.
Hint: Distinguish the cases $\lambda = 0$ and $\lambda \neq 0$.

Solution:
First, let's notice that it is enough to show one direction only, for instance the statement "if λ is an eigenvalue of AB, then λ is also an eigenvalue of BA". Indeed, if this implication is proved, the inverse one is also proved by symmetry (calling $B = A$ and $A = B$). We consider two different cases.

1. $\lambda = 0$.
   $\lambda = 0$ is an eigenvalue of AB if and only if $\det(AB) = 0 = \det(A)\det(B)$. This implies that either $\det A = 0$ or $\det B = 0$. But then we also have $\det(BA) = 0$, and this can be true if and only if $\lambda = 0$ is an eigenvalue of BA.

2. $\lambda \neq 0$.
   If $\lambda \neq 0$ is an eigenvalue of AB, then

$$\text{there is a } \mathbf{v} \neq \mathbf{0} \quad \text{such that} \quad AB\mathbf{v} = \lambda\mathbf{v} \neq \mathbf{0}. \tag{1}$$

Consider $\mathbf{w} = B\mathbf{v}$. Then $\mathbf{w} \neq \mathbf{0}$ because $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \notin \ker(B)$ by hypothesis (if $\mathbf{v} \in \ker(B)$ then $AB\mathbf{v} = \mathbf{0}$ which is in contradiction with the hypothesis that $\mathbf{v}$ is an eigenvector associated with an eigenvalue $\lambda \neq 0$). But then

$$BA\mathbf{w} = B(AB\mathbf{v}) = B(\lambda\mathbf{v}) = \lambda B\mathbf{v} = \lambda\mathbf{w} \tag{2}$$

thus λ is an eigenvalue of BA, too.

Ex 10.7 (Some statements about eigenvalues and eigenvectors)

1. Show that if λ is an eigenvalue of an invertible matrix A then λ^{-1} is an eigenvalue of A^{-1}.
   Hint: use a non zero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$.

2. Show that A and A^T have same eigenvalues.
   Hint: Find a relation between $\det(A - \lambda I)$ and $\det(A^T - \lambda I)$.

3. Let A be an $n \times n$ matrix such that the sum of each row's elements is equal to the same number r.

   (i) Show that r is an eigenvalue of A.
   Hint: find an eigenvector.

   (ii) *[This is a harder and therefore not exam-relevant but still fun problem:]*
   Show that if all the elements in A are in addition positive then the absolute value of any other eigenvalue is less than r.
   Hint: You will need to use the triangle inequality.

4. Let A be an $n \times n$ matrix such that the sum of each column's elements is equal to the same number c. Show that c is an eigenvalue of A.
   Hint: Use the results from the previous parts.

Solution:

1. If λ is an eigenvalue of A, then there is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. By successive steps we deduce from this

   $$A\mathbf{x} = \lambda\mathbf{x} \implies A^{-1}A\mathbf{x} = A^{-1}\lambda\mathbf{x} \implies \mathbf{x} = \lambda A^{-1}\mathbf{x} \implies A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}.$$

   Which means that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

2. As $\det(A - \lambda I) = \det(A - \lambda I)^T$ and $I = I^T$, we have $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$. Thus the characteristic polynomials of A and A^T are the same and those two matrices have the same eigenvalues.

3. (i) Let $\mathbf{v}$ be a $\mathbb{R}^n$ vector in which all the components are equal to 1. Then we have $A\mathbf{v} = r\mathbf{v}$ and thus r is an eigenvalue of A.

   (ii) Let λ be an eigenvalue of A; then there is a non-zero vector v such that $A\mathbf{v} = \lambda\mathbf{v}$:

   $$\begin{array}{rcl} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n & = & \lambda v_1 \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n & = & \lambda v_2 \\ \vdots & \vdots & \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n & = & \lambda v_n \end{array}$$

Consider M, $1 \leq M \leq n$, such that $|v_M| = \max(|v_i|)$, i.e., the entry of v with the biggest modulus. Sinve $v \neq 0$, we know that $v_M \neq 0$ and therefore the M^{th} line of the above system can be written as:

$$\lambda = a_{M1}\frac{v_1}{v_M} + a_{M2}\frac{v_2}{v_M} + \cdots + a_{Mn}\frac{v_n}{v_M}$$

As $\left|\frac{v_i}{v_M}\right| \leq 1$ for every i and as $a_{Mi} \geq 0, \forall i$, we have :

$$\begin{aligned}
|\lambda| &= \left|a_{M1}\frac{v_1}{v_M} + a_{M2}\frac{v_2}{v_M} + \cdots + a_{Mn}\frac{v_n}{v_M}\right| \\
&\leq a_{M1}\left|\frac{v_1}{v_M}\right| + a_{M2}\left|\frac{v_2}{v_M}\right| + \cdots + a_{Mn}\left|\frac{v_n}{v_M}\right| \\
&\leq a_{M1} + a_{M2} + \cdots + a_{Mn} = r.
\end{aligned}$$

4. In the matrix A^T the sum of the elements of each rows is equal to the same number c. By the preceding point c is an eigenvalue of A^T and by the second point of this exercise c is an eigenvalue of A.

Ex 10.8 (Even more eigenvalues and eigenvectors)

Consider the matrices

$$A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}.$$

For each of the matrices A and B: find its characteristic polynomial, its eigenvalues as well as their eigenvectors.

Hint: The characteristic polynomial of B is of degree 3 with no simple structure. To guess a root, you can use the rational root theorem: for a polynomial with **rational coefficients** $a_0 \ldots, a_n \in \mathbb{Q}$ any rational root (if it exists) has to be of the form a/b, where a divides the constant coefficient a_0 and b divides the leading coefficient a_n.

Solution:

The eigenvalues of A are the roots of its characteristic polynomial

$$P(\lambda) = \det(\lambda I - A) = \lambda^2 - 6\lambda + 8.$$

Thus we obtain the two roots $\lambda_1 = 2$ and $\lambda_2 = 4$. Since A is a 2×2 matrix and has 2 distinct eigenvalues, its eigenspaces are 1-dimensional (since eigenvectors corresponding to different eigenvalues are linearly independent).

The eigenvectors corresponding to $\lambda = 2$ satisfy $A\mathbf{v_1} = \mathbf{2v_1}$ and are of the form

$$\mu \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in \mathbb{R}.$$

The eigenvectors corresponding to $\lambda = 4$ satisfy $A\mathbf{v_2} = \mathbf{4v_2}$ and are of the form

$$\mu \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \mu \in \mathbb{R}.$$

As for B, its characteristic polynomial is

$$\det(\lambda I - B) = \begin{vmatrix} \lambda & 4 & 6 \\ 1 & \lambda & 3 \\ -1 & -2 & \lambda - 5 \end{vmatrix} = \lambda \cdot \begin{vmatrix} \lambda & 3 \\ -2 & \lambda - 5 \end{vmatrix} - 1 \cdot \begin{vmatrix} 4 & 6 \\ -2 & \lambda - 5 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 4 & 6 \\ \lambda & 3 \end{vmatrix}$$
$$= \lambda \cdot (\lambda^2 - 5\lambda + 6) - (4\lambda - 8) - (-6\lambda + 12) = \lambda^3 - 5\lambda^2 + 8\lambda - 4.$$

Using the rational root theorem, we look for a rational root that then has to be of the form a/b, where a divides (-4) and b has to divide 1 (which implies that $b = \pm 1$). Thus we have the possibilities $-4, -2, -1, 1, 2, 4$. A quick test shows that 1 is a zero, so we can factor

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(a\lambda^2 + b\lambda + c)$$

and a comparison of the coefficients yields that $a = 1$, $b = -4$ and $c = 4$. The remaining polynomial factors as

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2,$$

so that it total

$$\det(\lambda I - B) = (\lambda - 1)(\lambda - 2)^2.$$

Hence the eigenvalues of B are $\lambda_1 = 1$ and $\lambda_2 = 2$.
Solving the equation system $B\mathbf{v}_1 = 1 \cdot \mathbf{v}_1$, we find that the eigenvectors associated to the eigenvalue 1 are the vectors of the form

$$\mu \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \mu \in \mathbb{R}.$$

Solving the equation system $B\mathbf{v}_2 = 2 \cdot \mathbf{v}_2$, we find that the eigenvectors associated to the eigenvalue 2 are the vectors of the form

$$\mu \cdot \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} + \kappa \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \mu, \kappa \in \mathbb{R}.$$

Ex 10.9 (Who comes up with these titles?)
Let A and B be the following matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

For each matrix find out the characteristic polynomials, eigenvalues and the corresponding eigenvectors in $\mathbb{R}$.

Solution:
The characteristic polynomial of A is

$$\chi_A(\lambda) = \det(\lambda I - A) = (\lambda - 1)[\lambda^2 - \lambda - 1],$$

which can be obtained by Laplace expansion along the first column. Thus the three roots are $\lambda_1 = 1, \lambda_2 = (1 + \sqrt{5})/2, \lambda_3 = (1 - \sqrt{5})/2$. Since A is a 3×3 matrix and there are 3 distinct eigenvalues, each eigenspace is 1-dimensional. Eigenvectors corresponding to λ_1 are of the form

$$\mu \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mu \in \mathbb{R}.$$

Eigenvectors corresponding to λ_2 are of the form

$$\mu \cdot \begin{bmatrix} -2 \\ -2 \\ 1-\sqrt{5} \end{bmatrix}, \mu \in \mathbb{R}.$$

Eigenvectors corresponding to λ_3 are of the form

$$\mu \cdot \begin{bmatrix} -2 \\ -2 \\ 1+\sqrt{5} \end{bmatrix}, \mu \in \mathbb{R}.$$

The characteristic polynomial of B is

$$\chi_B(\lambda) = \det(\lambda I - B) = (\lambda - 1)^2,$$

with the root $\lambda = 1$, $\lambda \in \mathbb{R}$. Then B has only one eigenvalue of algebraic multiplicity equal to 2. Eigenvectors corresponding to λ are of the form

$$\mu \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mu \in \mathbb{R},$$

i.e., the eigenspace is 1-dimensional, meaning that the geometric multiplicity is 1.

Ex 10.10 (Rotation matrices 2×2**)**

Let α be an angle in $[0, 2\pi)$ and define the matrix R_α by:

$$R_\alpha := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

(a) Verify computationally that the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ induced by R_α (i.e. $f(v) = R_\alpha v$ for all $v \in \mathbb{R}^2$) is the counter-clockwise rotation around the origin by angle α.

Hint: Write $v = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ for some angel $\theta \in [0, 2\pi)$ (i.e. express v in polar coordinates), then compute $R_\alpha v$ and simplify using angle sum theorems of trigonometric functions. Then explain (e.g. in words and a drawing) why your computations show that f is the desired rotation.

(b) Compute all eigenvalues of R_α.

(c) For each eigenvalue, find at least one eigenvector of R_α. (For the case of non-real eigenvalues, this is not exam-relevant and hence not a mandatory exercise.)

Solution: a) Recall the double angle formulae:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

Thus, we have

$$R_\alpha v = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ -\sin\alpha\cos\theta + \cos\alpha\sin\theta \end{pmatrix} = \begin{pmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{pmatrix}.$$

Hence, as v is the unit vector in $\mathbb{R}^2$ with angle θ with respect to the x-axis in the counter-clockwise direction, applying R_α corresponds to rotating v by a further α angle in the counter-clockwise direction.

b) The characteristic polynomial of R_α is

$$\begin{aligned}\chi_{R_\alpha}(\lambda) = \det(\lambda I - R_\alpha) &= \det\begin{pmatrix}\lambda - \cos\alpha & \sin\alpha \\ -\sin\alpha & \lambda - \cos\alpha\end{pmatrix} \\ &= (\lambda - \cos\alpha)^2 + \sin^2\alpha \\ &= \lambda^2 - 2\cos\alpha\lambda + \cos^2\alpha + \sin^2\alpha \\ &= \lambda^2 - 2\cos\alpha\lambda + 1\end{aligned}$$

where we used the fact that $\cos^2\alpha + \sin^2\alpha = 1$. Thus, by the quadratic formula, χ_{R_α} has roots

$$\frac{1}{2}(2\cos\alpha \pm 2\sqrt{\cos^2\alpha - 1}) = \cos\alpha \pm \sqrt{-(1-\cos^2\alpha)} = \cos\alpha \pm i\sin\alpha$$

which are the eigenvalues of R_α.

c) Firstly, we observe that for $\alpha = k\pi$, $R_\alpha = (-1)^k I$ with the eigenvalue $(-1)^k$. In this case, it is clear that e_1, e_2 are suitable eigenvectors.
Now, for $\alpha \neq k\pi$, we want to find vectors $v \in \mathbb{R}^2$ such that

$$R_\alpha v = (\cos\alpha \pm i\sin\alpha)v.$$

Namely,

$$\begin{aligned}0 &= R_\alpha v - (\cos\alpha \pm i\sin\alpha)v \\ &= \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}v_1 \\ v_2\end{pmatrix} - \begin{pmatrix}\cos\alpha \pm i\sin\alpha & 0 \\ 0 & \cos\alpha \pm i\sin\alpha\end{pmatrix}\begin{pmatrix}v_1 \\ v_2\end{pmatrix} \\ &= \begin{pmatrix}\mp i\sin\alpha & -\sin\alpha \\ \sin\alpha & \mp i\sin\alpha\end{pmatrix}\begin{pmatrix}v_1 \\ v_2\end{pmatrix} = i\sin\alpha\begin{pmatrix}\mp 1 & i \\ -i & \mp 1\end{pmatrix}\begin{pmatrix}v_1 \\ v_2\end{pmatrix}.\end{aligned}$$

Hence, by dividing both sides by $i\sin\alpha$ (recall that we assumed $\alpha \neq k\pi$ so $\sin\alpha \neq 0$), it suffices to find solutions to the following homogeneous system of equations

$$\begin{pmatrix}\mp 1 & i \\ -i & \mp 1\end{pmatrix}\begin{pmatrix}v_1 \\ v_2\end{pmatrix} = 0.$$

This has solution space

$$\mathcal{S} = \left\{\begin{pmatrix}v_1 \\ \mp iv_1\end{pmatrix} : v_1 \in \mathbb{R}\right\}$$

Thus, taking $v_1 = 1$, the eigenvalue $\cos\alpha \pm i\sin\alpha$ has corresponding eigenvector

$$\begin{pmatrix}1 \\ \mp i\end{pmatrix}.$$

Ex 10.11 (Multiple choice and True/False questions)

a) (i) Let the matrix

$$A = \begin{bmatrix}-4 & 2 & -2 \\ 4 & -6 & \alpha \\ 5 & \beta & 7\end{bmatrix}$$

with the parameters $\alpha \in \mathbb{R}$ et $\beta \in \mathbb{R}$.

So, for all $\alpha \in \mathbb{R}$, the number -2 is an eigenvalue of the matrix A if

$(A)\quad \beta = 4 \qquad (B)\quad \beta = -5 \qquad (C)\quad \beta = -3 \qquad (D)\quad \beta = -2$

(ii) Let the matrix

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 5 & \alpha & 2 \\ \beta & 1 & 4 \end{bmatrix},$$

with the parameters $\alpha \in \mathbb{R}$ et $\beta \in \mathbb{R}$.

So, for all $\alpha \in \mathbb{R}$, the number 1 is an eigenvalue of the matrix A if

$$(A) \quad \beta = -3 \qquad (B) \quad \beta = 5/3 \qquad (C) \quad \beta = -2 \qquad (D) \quad \beta = 3$$

b) Decide whether the following statements are always true or if they can be false.

(i) If v_1 and v_2 are linearly independent eigenvectors of a matrix A, then they correspond to different eigenvalues.

(ii) The sum of two eigenvectors of a matrix is again an eigenvector.

(iii) If A is invertible and v is an eigenvector of A, then v is also an eigenvector of A^{-1}.

(iv) The eigenvalues of a matrix are the diagonal elements.

(v) Elementary row operations do not change the eigenvalues of a matrix.

(vi) A square matrix is invertible if and only if 0 is not an eigenvalue of A.

Solution:

a) (i) (B): To see this, insert consider the matrix $-2I_3 - A$ and compute its determinant which is $2(-\alpha+4)(\beta+5)$. This is zero for all α if and only if $\beta = -5$.

(ii) (D): In this case the determinant of $I_3 - A$ is $(-\beta+3)(2+3-3\alpha)$.

b)

(i) **FALSE**: Consider the matrix $A = I_n$, for which all vectors are eigenvectors with eigenvalue 1.

(ii) **FALSE**: Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, which has the eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, but $v_1 + v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is not an eigenvector.

(iii) **TRUE**: If A is invertible and v is an eigenvector with eigenvalue λ, then $\lambda \neq 0$ (otherwise $\ker(A)$ contains a non-zero vector) and $Av = \lambda v$. Multiplying this equation by A^{-1} yields $v = \lambda A^{-1} v$, so dividing by λ yields that v is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

(iv) **FALSE**: Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is not invertible and thus has the eigenvalue 0, which is not a diagonal element. The other eigenvalue is 2 (with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$), which is also not on the diagonal. However, the statement is true for triangular matrices.

(v) **FALSE**: We know that any invertible matrix can be transformed by elementary row operations into the identity matrix. However, the identity matrix only has the eigenvalue 1, which is not true for instance for the invertible matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(vi) **TRUE:** This was shown in the course.