MATH-111(en) FALL 2024
Linear Algebra Annina Iseli

MINI SOLUTIONS for Homework 9

Ex 9.1 (Column, row and kernels)
Find the dimensions of the column space, row space, and kernel of the following matrix.

$$B = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix}$$

Solution:
The column and row space has dimension 3 and the kernel has dimension 2.

Ex 9.2 (A subspace)
Find out the dimension of the subspace H defined as:

$$H = \left\{ x \text{ in } \mathbb{R}^4 \text{ such that } x = \begin{pmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{pmatrix} \text{ where } a, b, c \text{ and } d \text{ are real scalars} \right\}.$$

Solution: $\dim(H) = 3$

Ex 9.5(Change of basis matrices)
Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$, and consider the following basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

Find the change of basis matrices $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ and $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$.

Solution:

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \begin{pmatrix} -3 & 2 & 6 \\ 2 & -1 & -3 \\ -2 & 1 & 4 \end{pmatrix}.$$

Ex 9.6 (Changing coordinates)
Consider the following bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ Then determine $[v]_\mathcal{C}$ for $[v]_\mathcal{B} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } [v]_\mathcal{C} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Ex 9.7 (More basis changes)
Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be two bases of a vector space V. Assume that $b_1 = 6c_1 - 2c_2$ and $b_2 = 9c_1 - 4c_2$.

(a) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

(b) Find $[x]_\mathcal{C}$ for $x = -3b_1 + 2b_2$. Use the result from (a).

Let $\mathcal{A} = \{a_1, a_2\}$ and $\mathcal{D} = \{d_1, d_2\}$ be two bases of $\mathbb{R}^2$.

$$a_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad d_1 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

(c) Find the change of basis matrix $\underset{\mathcal{D}\leftarrow\mathcal{A}}{P}$

(d) Find the change of basis matrix $\underset{\mathcal{A}\leftarrow\mathcal{D}}{P}$

Solution:

(a) $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix}$ (b) $[x]_\mathcal{C} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ (c) $\underset{\mathcal{D}\leftarrow\mathcal{A}}{P} = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}$ (d) $\underset{\mathcal{A}\leftarrow\mathcal{D}}{P} = \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$

Ex 9.8 (Basis change for polynomials)
In $\mathbb{P}_2$, find out the change of base matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then write out the coordinates of the vector $x(t) = -1 + 2t$ in the basis $\mathcal{B}$.

Solution:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{pmatrix} \text{ and } [x]_\mathcal{B} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

Ex 9.10 (Finding the matrix of a linear transformation - warm up)
(a) Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be defined by

$$T(a_0 + a_1x + a_2x^2) = a_0 + a_2 + (2a_1 + a_2)x + (2a_1 + a_2)x^3.$$

Find a basis for $\mathrm{Ker}(T)$. Moreover: is $p(x) = 5x^2 - 5$ in $\mathrm{Ran}(T)$? Is it in $\mathrm{Ker}(T)$?

(b) Let $T : \mathbb{P}_3 \to \mathbb{R}^{2\times 3}$ be defined by

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = \begin{pmatrix} a_1 + a_2 & a_2 + a_3 & a_3 \\ a_2 + a_3 & 0 & a_0 \end{pmatrix}$$

Find the matrix A of T relative to the standard bases of $\mathbb{P}_3$ and $\mathbb{R}^{2\times 3}$. Then find a basis for $\operatorname{Ker}(A)$, $\operatorname{Col}(A)$ and $\operatorname{Row}(A)$. Also find a basis for $\operatorname{Ker}(T)$ and $\operatorname{Ran}(T)$.

Solution:

(a) $\mathcal{B}_{\operatorname{Ker}(T)} = \{-1 - \frac{1}{2}x + x^2\}$

(b) $\mathcal{B}_{\operatorname{Ker}(T)} = \emptyset$ (or, $\operatorname{Ker}(T)$ does not have a basis) and

$$\mathcal{B}_{\operatorname{Ran}(T)} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$