MATH-111(en) FALL 2024
Linear Algebra Annina Iseli

MINI SOLUTIONS for Homework 8

Ex 8.1 (A family of bases)
Find all $b \in \mathbb{R}$ such that the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ b \\ 0 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$.

Solution: $B = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ if and only if $b \neq 4$.

Ex 8.2 (Basis or not?)
Determine if

$$\{1 + t^2, 1 - t, 2 - 4t + t^2, 6 - 18t + 9t^2 - t^3\}$$

is a basis for $\mathbb{P}_3$.

Solution: They do form a basis.

Ex 8.3 (Bases of column and null spaces)
Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

(a) Find a basis for the column space of A.

(b) Find a basis for the null space of A.

Solution: Many solutions are possible as the basis of a vector space is not unique. Though by using the technique shown in class, you likely ended up finding the following solution:

(a)

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix} \right\} \text{ is a basis of } \mathrm{Col}(A).$$

(b)

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis of } \mathrm{Ker}(A).$$

Ex 8.7 (Representing a vector in a different basis)
Let $\mathcal{B} = \{b_1, b_2, b_3\}$ be the basis of $\mathbb{R}^3$ with

$$b_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the vector $u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, determine $[u]_{\mathcal{B}}$.

Moreover, find the vector w such that $[w]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

Solution:

$$[u]_{\mathcal{B}} = \frac{1}{4} \begin{pmatrix} 5 \\ 3 \\ -9 \end{pmatrix}, \qquad w = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}.$$

Ex 8.9 (New coordinates for polynomials)
Consider the basis $\mathcal{B} = \{p_1, p_2, p_3\}$ of $\mathbb{P}_2$ with

$$p_1(t) = 1 + t + t^2, \quad p_2(t) = 2t - t^2, \quad p_3(t) = 2 + t - t^2.$$

Determine $[t]_{\mathcal{B}}$ and $[1 + t^2]_{\mathcal{B}}$.
Hint: Write the $\mathcal{B}$-coordinates of p_1, p_2 and p_3 and the polynomials t and $1 + t^2$ for the basis $\mathcal{B} = \{1, t, t^2\}$ and then solve the corresponding linear systems.

Solution:

$$[t]_{\mathcal{B}} = \frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \qquad [1 + t^2]_{\mathcal{B}} = \frac{1}{7} \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$