MATH-111(en) FALL 2024
Linear Algebra Annina Iseli

MINI SOLUTIONS for Homework 7

Ex 7.2 (Is it a vector space?)

For each of the following sets (equipped with the obvious addition and scalar multiplication), decide whether it is a vector space and prove your result.

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \ x = 0 \right\}, \quad B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \ y = 1 \right\}, \quad C = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \ z = y \right\}$$

$$D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \ x, y, z \in \{0, -1, 1\} \right\}, \quad E = \left\{ f : \mathbb{R}^3 \to \mathbb{R}^3 \text{ linear} : \ f(e_1) = 0 \right\}$$

Solution: A,C, and E are vector spaces. B and D are not.

Ex 7.3 (Spaces of polynomials)

Let P_n be the vector space of polynomials of degree less than or equal to n. Determine if each of the following sets is a subspace of P_n for a given n. (You may take for granted that P_n is a vector space.)

a) The set of polynomials of the form $p(t) = at^2$ where a is an arbitrary real number.

b) The set of polynomials of the form $p(t) = a + t^2$ where a is an arbitrary real number.

c) The set of polynomials of the form $p(t) = c_1t^3 + c_2t^2 + c_3t + c_4$, where c_1, c_2, c_3 and c_4 are non-negative integers.

d) The set of polynomials in P_n that satisfy $p(0) = 0$.

Solution: (a) and (d) are subspaces. (b) and (c) are not.

Ex 7.6 (The only finite subspaces is $\{0_v\}$.)

Let V be a vector space and 0_V its zero element. Prove that $\{0_V\}$ is the only subspace of V that consists of only finitely many elements.

Hint:

Let H be a subspace of V that is not equal to $\{0_v\}$. Let $x \in H$ be a vector that is not 0_V. Try to prove the following claim and explain, why it solves Exercise 7.6.

Claim: Let $k_1, k_2 \in \mathbb{R}$. If $k_1 \neq k_2$ then $k_1x \neq k_2x$.
(In other words: for each k, kx is a different vector in H.)

Ex 7.7 (Column space and kernel)

(a) Does $v = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ lie in the column space of $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{pmatrix}$? Does it lie in its kernel?

(b) Let $B = \begin{pmatrix} 1 & 2 & -3 \\ 4 & -1 & 0 \\ 0 & -3 & 4 \end{pmatrix}$. Find a nonzero vector $u \in \mathrm{Col}(B)$ and a nonzero vector $v \in \mathrm{Ker}(B)$. Is there a nonzero vector that lies in both $\mathrm{Col}(B)$ and $\mathrm{Ker}(B)$?

(c) Express the kernel of the following matrix in parametric vector form:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

Solution:

(a) v lies in the column space but not the kernel.

(b) Omitted.

(c)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ -t \\ s \\ t \end{pmatrix} = s \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Ex 7.8 (Column space and kernel)
(a) Consider

$$w = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 & -5/2 \\ -3 & -2 & 4 \\ 2 & 4 & -4 \end{pmatrix}.$$

Find out if w is in $\mathrm{Col}\, A$, in $\mathrm{Ker}\, A$, or both.

(b) Find bases for the kernel, the column space, and the row space of $A = \begin{pmatrix} 1 & 1 & 5 & 1 \\ 2 & 4 & 14 & 4 \\ 2 & 3 & 12 & 3 \end{pmatrix}$

Solution:

(a): w is in $\mathrm{Ker}\, A$ as well as in $\mathrm{Col}\, A$.

(b): If you follow the steps of Example ($\star$) from class (Week 7). You will find obtain the following bases:

$$\mathcal{B}_{\mathrm{Ker}(A)} = \left\{ \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \mathcal{B}_{\mathrm{Col}(A)} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}, \quad \mathcal{B}_{\mathrm{Row}(A)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\},$$

Please be aware that different solutions are possible as no vector space has a unique basis.
If you need a repetition of the methods, the following youtube video goes over this exact example: `https://www.youtube.com/watch?v=AVWTlNkTNfw`