MATH-111(en) FALL 2024
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MINI SOLUTIONS for Homework 3

Ex 3.4 (The weekly linear system : matrix equations and linear (in)dependence)

For each of the matrix equations : (i) solve the equation. (ii) From the solution to the equation, deduce whether the columns of the coefficient matrices are linearly independent or linearly dependent.

$$\text{(a)} \begin{pmatrix} 2 & -5 & 8 \\ -2 & -7 & 1 \\ 4 & 2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & -3 & 7 \\ -2 & 1 & -4 \\ 1 & 2 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution :
a) $x_2 = 3/4x_3$ and $x_1 = -17/8x_3$.
b) $x_1 = x_2 = x_3 = 0$.

Ex 3.5 (Linear (in)dependence depending on a parameter)

For which values of $a \in \mathbb{R}$ are the following vectors linearly dependent ?

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ a \\ 2 \end{pmatrix}$$

Solution :
For $a = 3$ the vectors are linearly dependent.

Ex 3.6 (Vectors in the image of a linear transformation)

Consider the linear transformation (function) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1 - 2x_2 \\ -x_1 \\ x_1 - 2x_2 \end{pmatrix}.$$

1. Find $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Are there any more such vectors $\mathbf{x}$?
2. Is there an $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$?
3. Is there any vector $\mathbf{b}$ such that $T(\mathbf{x}) = \mathbf{b}$ has more than one solution ?

Solution :
Recall that for linear transformations an equation $T(x) = b$ corresponds to a linear system.

a) $\mathbf{x} = \begin{pmatrix} -1 \\ -3/2 \end{pmatrix}$.
b) Omitted.
c) Omitted.

Ex 3.8 (Representing linear transformations with matrices)
Find the matrices of the transformations T determined by the equations below.

1. $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2z - y \\ 3y - 2x \\ 4x - 3z \end{pmatrix}$.
2. $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$.
3. $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$.

   Hint: Express the vectors e_1, e_2, e_3 as linear combination of the vectors for which you know the image and then use linearity to compute what you need.

Solution :

1. $\begin{pmatrix} 0 & -1 & 2 \\ -2 & 3 & 0 \\ 4 & 0 & -3 \end{pmatrix}$
2. $\begin{pmatrix} 3 & 0 & 4 & -2 \end{pmatrix}$
3. $\begin{pmatrix} 4 & 1 & -9 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.