MATH-111(en) FALL 2024
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MINI SOLUTIONS for Homework 1

Mini solutions will only contain questions with numerical solutions. Namely, it will not include solutions to True/False, multiple choice nor *show that* type of questions.

Ex 1.1 (Solving linear systems I)

Solve the following systems of linear equations by finding the augmented matrix and using row elimination.

$$a)\ \begin{cases} x+y &= 5 \\ 2x-5y &= 4 \end{cases} \qquad b)\ \begin{cases} x+y+z &= 3 \\ x-y &= 0 \\ x &= -46 \end{cases} \qquad c)\ \begin{cases} 2x+3y+z &= 0 \\ x-y+z &= 1 \\ 3x+2y+2z &= 1 \end{cases}$$

Solution :

a)

$$x = \frac{29}{7}, \quad y = \frac{6}{7}$$

b)

$$\longrightarrow \ x = -46, \quad y = -46, \quad z = 95$$

c)There are infinitely many solutions with the solution space being

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = -\frac{4}{5}z + \frac{3}{5}, \quad y = \frac{1}{5}z - \frac{2}{5}, \quad z \in \mathbb{R} \right\}$$

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Ex 1.2 (Solving linear systems II)

Find the augmented matrix of the following linear systems and use it to solve them.

$$a)\ \begin{cases} w-x+z &= 1 \\ x-2y-z &= 0 \\ w+3y &= 2 \end{cases} \qquad b)\ \begin{cases} 2x-5y+4z &= 0 \\ x+y+z &= 0 \\ 4x-3y+6z &= 1 \end{cases} \qquad c)\ \begin{cases} x-2y+3z &= 1 \\ 2x-4y+6z &= 2 \\ -x+2y-3z &= -1 \end{cases}$$

Solution :

a) The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 1 & 0 & 3 & 0 & 2 \end{array}\right)$$

with the solution space being

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y = \frac{1}{5}, \quad x = \frac{2}{5} + z, \quad w = \frac{7}{5}, \quad z \in \mathbb{R} \right\}$$

b) The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & -5 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 4 & -3 & 6 & 1 \end{array}\right)$$

and there are no solutions.
c)

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & 2 \\ -1 & 2 & -3 & -1 \end{array}\right)$$

and the solutions space is

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \quad x = 2y - 3z + 1, \quad y, z \in \mathbb{R} \right\}$$

Ex 1.3 (Solving linear systems III)
Find the augmented matrix of the following linear systems and use it to solve them.

$$a) \begin{cases} w - x + y - z &= 1 \\ w + z &= 2 \end{cases} \qquad b) \begin{cases} x + y &= 0 \\ 3x + 5y &= 2 \\ 2x + 4y &= 2 \end{cases}$$

Solution :
a) The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{array}\right)$$

and the solutions space is

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x = y - 2z + 1, \quad w = -z + 2, \quad y, z \in \mathbb{R} \right\}$$

b) The augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 5 & 2 \\ 2 & 4 & 2 \end{array}\right)$$

and the solutions space is

$$\mathcal{S} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

Ex 1.4 (Linear systems with a parameter)
Determine for which $a \in \mathbb{R}$ the following system has no solution, a unique solution, or infinitely many solutions.

$$\begin{cases} x - 2y + 3z &= 2 \\ x + 3y - 2z &= 5 \\ 2x - y + az &= 1 \end{cases}$$

Solution :

$$\begin{cases} a = 3: & \text{no solutions} \\ a \neq 3: & \text{unique solution} : x = \frac{16}{5} + \frac{24}{5a-15}, \quad y = \frac{3}{5} - \frac{24}{5a-15}, \quad z = \frac{-24}{5a-15} \end{cases}$$

Ex 1.5 (Solvability of parameter-dependent systems)
Determine the values of h for which the following matrices are the augmented matrices of a consistent linear system. (A linear system is called *consistent* if it has at least one solution. It is *inconsistent* if there exists no solution.)

$$\text{(a)} \begin{pmatrix} 1 & -3 & h \\ -2 & 6 & -5 \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{pmatrix}.$$

Solution :
a) It is consistent if and only if $h = 5/2$.

b) It is consistent if and only if $h \neq 2$.

Ex 1.6 (Linear combinations)
a) For the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 11 \\ 16 \\ 21 \end{pmatrix}$, find $l, m \in \mathbb{R}$ such that $\mathbf{c} = l\mathbf{a} + m\mathbf{b}$.

b) Find all $a \in \mathbb{R}$ such that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 2 \end{pmatrix} \right\}$.

Solution :
a) $l = 3, m = 2$
b) $a = 2$ or $a = -1$.