MATH-111(en) FALL 2024
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MINI SOLUTIONS for Homework 14

Ex 14.1 (Orthogonal diagonalization)

Orthogonally diagonalize the matrices $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -3 & 0 & 0 \\ -3 & 12 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -3 & 12 \end{pmatrix}$

Solution: <u>Solution for A:</u>

$$A = UDU^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \cdot \left(\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\right).$$

<u>Solution for B:</u>

$$U = \begin{pmatrix} \frac{-1}{\sqrt{10}} & 0 & 0 & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & 0 & 0 & \frac{1}{\sqrt{10}} \\ 0 & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0 \\ 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 13 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad A = UDU^T$$

Ex 14.2 (Orthogonal diagonalization with some help)
Consider

$$A = \begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}, \quad \mathbf{v_1} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

1. Check that v_1 and v_2 are eigenvectors of A.
2. Orthogonally diagonalize the matrix A. (*Hint*: Make use of the fact that you already know two eigenvectors instead of just using the standard recipe for orthogonal diagonalization!)

Solution:

1. Omitted.
2. $$Q = \begin{pmatrix} -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{pmatrix}, \qquad D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{and } A = QDQ^T.$$

Ex 14.3 (Computing an SVD)
Find the singular value decomposition of each of the following matrices:

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution:

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Ex 14.4 (SVD with higher geometric multiplicity)
Find the singular value decomposition of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution:

$$A = U\Sigma V^T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & 0 & \frac{4}{5} & 0 \\ 0 & -\frac{4}{5} & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Ex 14.6 (Computing SVD from eigenvectors)
Let $A \in \mathbb{R}^{2\times 4}$, $w_1, w_2 \in \mathbb{R}^4$ be such that w_1, w_2 are eigenvectors of A^TA, and

$$w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \; w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \; Aw_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \; Aw_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Find matrices U, Σ and V such that A has singular value decomposition of the form

$$A = U\Sigma V^T.$$

Solution:

$$A = U\Sigma V^T = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{5}{2}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{3}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Ex 14.7 (Calculating $\exp(tA)$ and solving ODEs)
Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. (a) Compute $\exp(tA)$. (**Hint**: Diagonalize A.)

b) Solve the differential equation $x'(t) = A \cdot x(t)$ for each of the initial values:

$$(i)\ x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad (ii) \quad x(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Solution:

a) $\exp(tA) = \frac{1}{2} \begin{pmatrix} e^t + e^{3t} & e^{3t} - e^t \\ e^{3t} - e^t & e^t + e^{3t} \end{pmatrix}.$

b) i) $x(t) = \begin{pmatrix} -\exp(t) \\ \exp(t) \end{pmatrix}.$

ii) $x(t) = \begin{pmatrix} \exp(t) + 3\exp(3t) \\ 3\exp(3t) - \exp(t) \end{pmatrix}.$

Ex 14.8 (Solving ODEs) Solve the following system of differential equations :

$$\begin{cases} x_1'(t) &= 5x_1(t) - 4x_2(t) - 2x_3(t) \\ x_2'(t) &= -4x_1(t) + 5x_2(t) + 2x_3(t) \\ x_3'(t) &= -2x_1(t) + 2x_2(t) + 2x_3(t) \end{cases}$$

for the initial values $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 1$

Hint: transfer it into a suitable matrix form. Then before your start investing loads of time into computations, ask yourself whether the matrix looks familiar to you.

Solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} \frac{2}{9}(-e^{10t} + e^t) \\ \frac{1}{9}(2e^{10t} - 2e^t) \\ \frac{1}{9}(e^{10t} + 8e^t) \end{pmatrix}.$$

Ex 14.9 (A higher order ODE)

Consider the following differential equation: $y'''(t) + 4y''(t) - 4y'(t) = 0$

(a) Transform this ODE of order n into a system of ODEs of order 1 and write it in matrix-vector-form.

(b) Compute $\exp(tA)$ for $t \in \mathbb{R}$.

(c) Using the method of matrix exponentials, compute a solution $y(t)$ for the differential equation for the initial values: $y''(0) = y'(0) = 0$ and $y(0) = 1$?

Solution:

$$(a) \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & -4 \end{pmatrix},$$

$$(b) \quad \exp(tA) = S \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{(2\sqrt{2}-1)t} & 0 \\ 0 & 0 & e^{-2(1+\sqrt{2})t} \end{pmatrix} S^{-1} = \ldots$$

$$(c) \quad y(t) = 1.$$

Ex 14.11 (Diagonalization of a matrix exponential)
Let A be the matrix from Exercise 11.2 (see Homework 11). Diagonalize $\exp(tA)$ for $t \in \mathbb{R}$.

Solution:

$$\exp(tA) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Since we were asked to *diagonalize* $\exp(tA)$ (as opposed to *computing* it) we do not have to further simplify.