MATH-111(en) FALL 2024
Linear Algebra Annina Iseli

MINI SOLUTIONS for Homework 10

Ex 10.1 (Calculus with complex numbers)
a) Find the real and imaginary part of $z = \dfrac{1}{1+i} - \dfrac{2+3i}{2+i}$.

b) Calculate the modulus of the following complex numbers:

$$z_1 = i, \quad z_2 = 2 - 2i, \quad z_3 = -4i.$$

Solution:
a) $\mathrm{Re}(z) = -\frac{9}{10}$ and $\mathrm{Im}(z) = -\frac{13}{10}$.

b) $|z_1| = 1$, $|z_2| = \sqrt{8}$ and $|z_3| = 4$.

Ex 10.2 (The matrix of a linear map relative to two bases)
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 4x_1 + 3x_2 \\ 10x_1 - 8x_2 \\ x_1 + 2x_2 \end{pmatrix}.$$

Consider the bases $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$ and $\mathcal{C} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$. Give the matrix $M = \underset{\mathcal{C} \leftarrow \mathcal{B}}{[T]}$ that represents T going from $\mathcal{B}$ to $\mathcal{C}$; in other words, the matrix M such that $[T(\mathbf{v})]_{\mathcal{C}} = M \cdot [\mathbf{v}]_{\mathcal{B}}$.

Solution:

$$M = \begin{pmatrix} -2 & 5 \\ 1 & -4 \\ 3 & 2 \end{pmatrix}$$

Ex 10.3 (Finding a basis for an eigenspace)
Find a basis for the eigenspace of the eigenvalue $\lambda = 3$ of the matrix

$$B = \begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}.$$

Solution:

One possible choice of the basis for the eigenspace is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Ex 10.4 (Some eigenvalues and eigenvectors)
Consider the matrices

$$C = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix}.$$

Find the eigenvalues of both, and give an eigenvector for each eigenvalue.

Solution:
$\lambda_1 = 2$ and $\lambda_2 = 4$ are eigenvalues of C with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$\lambda = 3$ is the only eigenvalue for D with an eigenvector

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Ex 10.5 (More eigenvalues and eigenvectors)
Find the eigenvalues of

$$E = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and give an eigenvector for each.

Solution:
$\lambda_1 = -1, \quad \lambda_2 = 4, \quad \lambda_3 = 2$ are eigenvalues with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}.$$

Ex 10.8 (Even more eigenvalues and eigenvectors)
Consider the matrices

$$A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}.$$

Find the characteristic polynomials of A and B, their eigenvalues and the associated eigenvectors.
Hint: The characteristic polynomial of B is of degree 3 with no simple structure. To guess a root, you can use the rational root theorem: for a polynomial with **rational coefficients** $a_0 \ldots, a_n \in \mathbb{Q}$

any rational root (if it exists) has to be of the form a/b, where a divides the constant coefficient a_0 and b divides the leading coefficient a_n.

Solution:
A has characteristic polynomial
$$P(\lambda) = \lambda^2 - 6\lambda + 8.$$
and eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$ with corresponding eigenvectors
$$\mu \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in \mathbb{R} \text{ and } \mu \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \mu \in \mathbb{R}.$$
B has characteristic polynomial
$$P(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4.$$
The eigenvalues of B are $\lambda_1 = 1$ and $\lambda_2 = 2$ with corresponding eigenvectors
$$\mu \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \mu \in \mathbb{R},$$
and
$$\mu \cdot \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} + \kappa \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \mu, \kappa \in \mathbb{R}.$$

Ex 10.9 (Who comes up with these titles?)
Let A and B be the following matrices
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$
For each matrix find out the characteristic polynomials, eigenvalues and the corresponding eigenvectors in $\mathbb{R}$.

Solution:
A has eigenvalues $\lambda_1 = 1, \lambda_2 = (1+\sqrt{5})/2, \lambda_3 = (1-\sqrt{5})/2$ with corresponding eigenvectors
$$\mu \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mu \in \mathbb{R}, \mu \cdot \begin{bmatrix} -2 \\ -2 \\ 1-\sqrt{5} \end{bmatrix}, \mu \in \mathbb{R} \text{ and } \mu \cdot \begin{bmatrix} -2 \\ -2 \\ 1+\sqrt{5} \end{bmatrix}, \mu \in \mathbb{R}.$$
B has only eigenvalue $\lambda = 1$ with eigenvectors
$$\mu \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mu \in \mathbb{R},$$