MATH-111(en) FALL 2024
Linear Algebra Annina Iseli

Homework 9

Ex 9.1 (Column space, row space and kernel)

Find the dimensions of the column space, row space, and kernel of the following matrix.

$$B = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix}$$

Ex 9.2 (A subspace)

What is the dimension of the subspace $H \subset \mathbb{R}^4$ defined below?

$$H = \left\{ x \text{ in } \mathbb{R}^4 \text{ such that } x = \begin{pmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{pmatrix} \text{ where } a, b, c \text{ and } d \text{ are real scalars} \right\}.$$

Ex 9.3 (Row equivalent matrices)

Consider the matrices

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Show that A and B are row equivalent. Then, deduce

- the rank of A and $\dim \operatorname{Ker} A$
- a basis for each of the subspaces $\operatorname{Col} A$, $\operatorname{Row} A$, and $\operatorname{Ker} A$.

Ex 9.4 (Relating A and A^T in terms of linear systems)

Consider a matrix $A \in \mathbb{R}^{m \times n}$. Among the spaces $\operatorname{Row} A$, $\operatorname{Col} A$, $\operatorname{Ker} A$, $\operatorname{Row} A^T$, $\operatorname{Col} A^T$ and $\operatorname{Ker} A^T$, find out which are subspaces of $\mathbb{R}^n$, and which are subspaces of $\mathbb{R}^m$.
Then justify the following statements:

1. $\dim \operatorname{Row} A + \dim \operatorname{Ker} A = n$ (number of columns in A).
2. $\dim \operatorname{Col} A + \dim \operatorname{Ker} A^T = m$ (number of rows in A).
3. $Ax = b$ has a solution for every b in $\mathbb{R}^m$ if and only if $A^T x = 0$ only admits the trivial solution.

Ex 9.5(Change of basis matrices)

Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$, and consider the following basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

Find the change of basis matrices $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$.

Ex 9.6 (Changing coordinates)

Consider the following bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Then find $[v]_\mathcal{C}$ for $[v]_\mathcal{B} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Ex 9.7 (More basis changes)

Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be two bases of a vector space V. Assume that $b_1 = 6c_1 - 2c_2$ and $b_2 = 9c_1 - 4c_2$.

(a) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

(b) Find $[x]_\mathcal{C}$ for $x = -3b_1 + 2b_2$. Use the result from (a).

Let $\mathcal{A} = \{a_1, a_2\}$ and $\mathcal{D} = \{d_1, d_2\}$ be two bases of $\mathbb{R}^2$.

$$a_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad d_1 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

(c) Find the change of basis matrix $\underset{\mathcal{D}\leftarrow\mathcal{A}}{P}$

(d) Find the change of basis matrix $\underset{\mathcal{A}\leftarrow\mathcal{D}}{P}$

Ex 9.8 (Basis change for polynomials)

In $\mathbb{P}_2$, find out the change of base matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then write out the coordinates of the vector $x(t) = -1 + 2t$ in the basis $\mathcal{B}$.

Ex 9.9 (The trace of a matrix as linear map)

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. We define the trace of A by $\mathrm{Tr}(A) = a_{11} + \ldots + a_{nn}$, i.e., the sum of all diagonal elements.

a) Show that the map $\mathrm{Tr} : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a linear map.

b) Consider the case $n = 2$ and the ordered basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

while on $\mathbb{R}$ we consider the standard basis $\mathcal{Q} = \{1\}$. Compute the matrix B such that $[\mathrm{Tr}(A)]_\mathcal{Q} = B[A]_\mathcal{B}$ for all $A \in \mathbb{R}^{2\times 2}$.

Ex 9.10 (Finding the matrix of a linear transformation - warm up)

(a) Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be defined by

$$T(a_0 + a_1x + a_2x^2) = a_0 + a_2 + (2a_1 + a_2)x + (2a_1 + a_2)x^3.$$

Find a basis for $\mathrm{Ker}(T)$. Moreover: is $p(x) = 5x^2 - 5$ in $\mathrm{Ran}(T)$? Is it in $\mathrm{Ker}(T)$?

(b) Let $T : \mathbb{P}_3 \to \mathbb{R}^{2\times 3}$ be defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{pmatrix} a_1 + a_2 & a_2 + a_3 & a_3 \\ a_2 + a_3 & 0 & a_0 \end{pmatrix}$$

Find the matrix A of T relative to the standard bases of $\mathbb{P}_3$ and $\mathbb{R}^{2\times 3}$. Then find a basis for $\mathrm{Ker}(A)$, $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$. Also find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$.

Ex 9.11 (Finding the matrix of a linear transformation)

Let $T : \mathbb{R}^{2\times 2} \to \mathbb{P}_5$ be defined by

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 2x^5 - 3x^4 + 5x, \quad T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -x^2 + x + 1,$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -x^2 + x + 1, \quad T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -2x^4 + x^3 - x^2 + 1$$

Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$.

Hint: Try choosing a clever basis $\mathcal{B}$ for $\mathbb{P}_5$ instead of the standard basis. Computations will become a lot easier.

Ex 9.12 (True/False questions)

In the following, let A be an $m \times n$ matrix and $\mathcal{B}, \mathcal{C}$ bases of a vector space V. Decide whether the following statements are always true or if they can be false.

(i) $\mathrm{Row}(A) = \mathrm{Col}(A^T)$.

(ii) $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$.

(iii) $\dim \mathrm{Row}(A) + \dim \mathrm{Ker}(A) = n$.

(iv) There is a 6×9 matrix B such that $\dim \mathrm{Ker}(B) = 2$.

(v) If a set $\{v_1, \ldots, v_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

(vi) The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

(vii) If B is any echelon form of A, and if B has three nonzero rows, then the first three rows of A form a basis for $\mathrm{Row}\, A$.

(viii) The dimension of the kernel of A is the number of columns of A that are *not* pivot columns.

(ix) The row space of A^T is the same as the column space of A.

(x) The columns of the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$.

(xi) If $V = \mathbb{R}^n$ and $\mathcal{C}$ is the *standard* basis V, then $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced earlier.