MATH-111(en) Fall 2024
Linear Algebra Annina Iseli

Homework 8

Ex 8.1 (A family of bases)
Find all $b \in \mathbb{R}$ such that the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ b \\ 0 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$.

Ex 8.2 (Basis or not?)
Determine if

$$\{1 + t^2, 1 - t, 2 - 4t + t^2, 6 - 18t + 9t^2 - t^3\}$$

is a basis for $\mathbb{P}_3$.

Ex 8.3 (Bases of column space and kernel)
Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

(a) Find a basis for the column space of A.
(b) Find a basis for the kernel of A.
(c) What are the respective dimensions of the range and kernel of A?

Ex 8.4 (Kernel and range)

(a) Let $T : \mathbb{P}_3 \to \mathbb{P}$ be the linear transformation defined by

$$T(p) = p'$$

Find $\text{Ker}(T)$ and $\text{Ran}(T)$, as well as bases for each of them.

(b) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(p(t)) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix},$$

where p' is the derivative of p. Find bases for $\text{Ker}(T)$ and $\text{Ran}(T)$.

Ex 8.5 (A basis calculation)
Find a basis for the space spanned by the following vectors:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}.$$

Ex 8.6 (Getting acquainted with kernel and column space)
Let A be an $m \times n$ matrix, B an $n \times k$ matrix such that $\mathrm{Ker}(A) \cap \mathrm{Col}(B) = \{0\}$, and $\mathcal{B} = \{b_1, \cdots, b_k\}$ a basis of $\mathrm{Col}(B)$.
Show that $\mathcal{C} = \{Ab_1, \cdots, Ab_k\}$ is a basis of $\mathrm{Col}(AB)$.

Ex 8.7 (Representing a vector in a different basis)
Let $\mathcal{B} = \{b_1, b_2, b_3\}$ be the basis of $\mathbb{R}^3$ with

$$b_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the vector $u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, determine $[u]_{\mathcal{B}}$.

Moreover, find the vector w such that $[w]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

Ex 8.8 (More coordinate calculations)
We define:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ -7 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad [x]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}. \quad \text{and} \quad y = \begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix}$$

Find the vector x (*i.e.* its coordinates in the standard basis) and find $[y]_{\mathcal{B}}$.

Ex 8.9 (New coordinates for polynomials)

Determine $[t]_{\mathcal{B}}$ and $[1+t^2]_{\mathcal{B}}$ for the basis $\mathcal{B} = \{p_1, p_2, p_3\}$ of $\mathbb{P}_2$ where

$$p_1(t) = 1 + t + t^2, \quad p_2(t) = 2t - t^2, \quad p_3(t) = 2 + t - t^2.$$

(Hint: Write the $\mathcal{B}$-coordinates of p_1, p_2 and p_3 and the polynomials t and $1+t^2$ for the basis $\mathcal{B} = \{1, t, t^2\}$ and then solve the corresponding linear systems.)

Ex 8.10 (More polynomial calculations)

(a) Show that the set $F = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$.

(b) Find the coordinates vector of $f(t) = 1 + 4t + 7t^2$ in the basis F.

Ex 8.11 (Dimension of the kernel) Let $A \in \mathbb{R}^{n \times n}$ and assume that the dimension of $\mathrm{Ker}(A) = 1$. Can $\dim \mathrm{Ker}(A^2)$ be equal to 0? Can it be equal to 1 or 2? Can it be larger than 2?

Tipp: Start by trying to come up with a few simple examples of matrices A for which $\dim \mathrm{Ker}(A) = 1$ and check what $\dim \mathrm{Ker}(A^2)$ is.

(Be aware: the last question is more tricky than the others and is not a potential exam problem.)

Ex 8.12 (True/False questions)
Decide whether the following statements are always true or if they can be false.

(i) If $V = \text{Span}(v_1, \dots, v_k)$, then $\{v_1, \dots, v_k\}$ is a basis of V.

(ii) A spanning set of maximal size is a basis.

(iii) Suppose the matrix B is an echelon form of the matrix A. Then the pivot columns of B form a basis for $\text{Col}(A)$.

(iv) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

(v) A linearly independent set in a subspace H is a basis for H.

(vi) If V is a vector space and $\mathcal{B}$ a basis with n elements, then $[x]_{\mathcal{B}}$ is a vector in $\mathbb{R}^n$.

(vii) If V is a vector space with a finite basis $\mathcal{B}$ and $P_{\mathcal{B}}$ is the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis, then $[x]_{\mathcal{B}} = P_{\mathcal{B}}\, x$ for all $x \in V$.