MATH-111(en) FALL 2024
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Homework 12

Ex 12.1 (Diagonalizability)

a. Let A be a 3×3 matrix satisfying $A^4 = I_3$. Can 0 be an eigenvalue of A?

b. Let A be a 3×3 with characteristic polynomial equal to $\chi_A(\lambda) = (\lambda - 1)^2(\lambda + 1)$. Which of the following statements is true?

☐ A must be diagnoalizable.

☐ $\sigma(A) = \{-1, 1\}$

☐ A cannot be diagonalizable.

☐ In case A is diagonalizable, then there exist linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$ each satisfying $Av_i = -v_i$.

Ex 12.2 (Inner product calculations)

Let

$$u = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \qquad v = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}.$$

a) Calculate $u \cdot u$, $v \cdot v$, $u \cdot v$, $\|u\|$, and $\|v\|$.

b) Normalize u and v (i.e., find a unit vector with the same direction).

c) Find the distance between u and v, and find the cosine of the angle between them.

d) Find a basis of the space orthogonal to the plane spanned by u and v.

Ex 12.3 (An orthogonal basis)

Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}, \qquad u = \begin{pmatrix} 10 \\ 4 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

Show that $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^3$ and determine $[u]_{\mathcal{B}}$ and $[v]_{\mathcal{B}}$, i.e. represent them in the basis $\mathcal{B}$.

Ex 12.4 (Another orthogonal basis)

Consider the vectors

$$u = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \quad x = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

(a) Show that $\{u, v, w\}$ is an orthogonal basis of $\mathbb{R}^3$.

(b) Write the vector x as a linear combination of u, v and w.

Ex 12.5 (Properties of the orthogonal complement)
Let $W \subset \mathbb{R}^n$ be a subspace and $W^\perp$ be its orthogonal complement. Show the following statements:

- Lemma 6.2:
  (i) $W^\perp$ is a subspace of $\mathbb{R}^n$. Moreover, $W \cap W^\perp = \{0\}$.
  (ii) If $\mathcal{B}$ spans W, then $W^\perp = \{z \in \mathbb{R}^n : z \cdot b = 0 \quad \forall b \in \mathcal{B}\}$.
- Theorem 6.4: $\dim(W^\perp) = n - \dim(W)$.
  Hint: Let $b_1, ..., b_k$ be a basis of W and M the matrix whose rows are the b_i. Check that $W^\perp = \mathrm{Ker}(M)$ and use Theorem 6.3.

Ex 12.6 ($F^T F$ vs. FF^T for matrices with orthogonal columns)
Consider the matrix

$$F = \begin{pmatrix} 1 & 2 \\ -4 & 1/2 \end{pmatrix}.$$

Compute $F^T F$ and FF^T. Are these two matrices equal ?

Ex 12.7 (Orthogonality and projections) Prove the following statements about orthogonality and projections:

(i) Every orthogonal set that does not contain the zero vector is independent.
(This implies that in particular orthonormal sets are independent.)

(ii) The orthogonal projection from $\mathbb{R}^n$ onto a linear subspace $W \subset \mathbb{R}^n$ is a linear map.

Ex 12.8 (Projection onto a subspace)
Let

$$u = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Determine the orthogonal projection $\mathrm{proj}_W(u)$ of u onto the subspace W spanned by v_1, v_2. Give it both in the basis $\mathcal{B} = \{v_1, v_2\}$ of W and in the standard basis of $\mathbb{R}^3$.

Ex 12.9 (The row space and the kernel)
Consider an $m \times n$ matrix A.

a) Prove that every vector x in $\mathbb{R}^n$ can be written uniquely as $x = p + u$ where p belongs to $\mathrm{Row}(A)$ and u belongs to $\mathrm{Ker}(A)$.

b) Afterwards, show that if the equation $Ax = b$ is consistent, then there is a unique p in $\mathrm{Row}(A)$ such that $Ap = b$.
Hint: For uniqueness, use Lemma 6.2 (b).

Ex 12.10 (Closest point in a column space)
Let A be the following matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

1. Show that the columns of A are an orthogonal set.
2. Write U, the matrix made of the normalized columns vectors of A.
3. Find the closest point to $y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathrm{Col}(U)$ and the distance from $b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ to $\mathrm{Col}(U)$.

Ex 12.11 (Distance to different subspaces)

Let

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

Compute the distance from u to the line spanned by v_1, and the distance from u to the plane spanned by v_1 and v_2.

Ex 12.12 (Multiple choice and True/False questions)

a) Let $A \in \mathbb{R}^{3\times 3}$. Which of the following sets of eigenvalues is possible?

$$\text{(A)} \quad \{1, 1+i, 2-i\}, \quad \text{(B)} \quad \{1, 2, 4i\}, \quad \text{(C)} \quad \{0, 3-i, 3+i\}, \quad \text{(D)} \quad \{i, 3-i, 3+i\}.$$

b) Decide whether the following statements are always true or if they can be false.

(i) Let $u, v, w \in \mathbb{R}^n$. If $u \cdot v = 0$ and $v \cdot w = 0$, then $u \cdot w \neq 0$.

(ii) Let $u, v \in \mathbb{R}^n$. If the distance between u and v equals the distance between u and $-v$, then u and v are orthogonal.

(iii) If $A \in \mathbb{R}^{n\times n}$, then $\mathrm{Col}(A) = \mathrm{Ker}(A)^\perp$.

(iv) Let W be a subspace of $\mathbb{R}^n$. If x is orthogonal to every element of a basis for W, then $x \in W^\perp$.

(v) If $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then $\|\lambda x\| = \lambda \|x\|$.

(vi) The orthogonal projection of u onto v is the same as the orthogonal projection of u onto av for any $a \neq 0$.

(vii) If W is a subspace of $\mathbb{R}^n$ and $u \in W$, then $\mathrm{proj}_W(u) = u$.

(viii) Let A be an $n \times n$ matrix. The columns of A form an orthonormal basis of $\mathbb{R}^n$ if and only if $\det(A) = 1$.

(ix) If $A^T A = I$, then A must be square.

(x) A square matrix has orthonormal columns if and only if it has orthonormal rows.

(xi) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

(xii) A matrix with orthonormal columns is an orthogonal matrix.

(xiii) For each $y \in \mathbb{R}^n$ and each subspace W of $\mathbb{R}^n$, the vector $y - \mathrm{proj}_W\, y$ is orthogonal to W.

(xiv) If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T y$ is the orthogonal projection of y onto the column space of U.