MATH-111(en) FALL 2024
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Homework 11

Ex 11.1 (Diagonalizable or not?)
Which of the following matrices are diagonalizable?

$$M_1 = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 6 & 0 \\ 1 & -2 & 2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Ex 11.2 (Diagonalization of a matrix)
Diagonalize the following matrix.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

If not stated otherwise, this means finding a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. In particular, you do not need to compute P^{-1}.

Ex 11.3 (Déjà vu?)
Diagonalize the following matrix.

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Ex 11.4 (More diagonalizability examples)
Consider

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}, \; B = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \; C = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}, \; D = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \quad \text{and } E = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}.$$

1. For each matrix find out the eigenvalues and the corresponding eigenvectors.
   Hint: For C and D the rational root theorem (see Ex. 10.8) helps to find the eigenvalues.
2. Find out which ones are diagonalizable.

Ex 11.5 (Powers of a diagonalizable matrix)
Let $A = PDP^{-1}$ with $P \in \mathbb{R}^{n\times n}$ invertible and $D \in \mathbb{R}^{n\times n}$ a diagonal matrix. Show that for any $k \in \mathbb{N}$ it holds that $A^k = PD^kP^{-1}$.
Remark: Powers of a diagonal matrix are easy to calculate. We have seen in the course that we just need to take the corresponding powers of the diagonal elements.

Ex 11.6 (Matrix representation of linear maps)
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 4z \\ 3x + 5y - 2z \\ x + y + 4z \end{pmatrix}.$$

Consider the ordered basis of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Find the matrix $M = \underset{\mathcal{B} \leftarrow \mathcal{B}}{[T]}$ that represents T in the basis $\mathcal{B}$.

Ex 11.7 (Another matrix representation)
Let the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined by:

$$T(\mathbf{p}) = \begin{pmatrix} p(0) \\ p(0) \\ p(2) \end{pmatrix} \quad \text{for any polynomial } \mathbf{p} \in \mathbb{P}_2.$$

a) Find the matrix A of the linear transformation T in terms of the standard basis of $\mathbb{P}_2$ and the standard basis of $\mathbb{R}^3$.

b) Using the matrix A, determine the kernel and image of T.

Ex 11.8 (Partial proof of Theorem 5.10)
Let $A \in \mathbb{R}^{n \times n}$ and $\sigma(A) = \{\lambda_1, ..., \lambda_k\}$ (distinct list of eigenvalues). Verify that:

(1) If A is diagonalizable, then for each i:
$\mathrm{multgeom}_A(\lambda_i) = \mathrm{multalg}_A(\lambda_i)$.

(2) A is diagonalziable if and only if $\sum_{i=1}^{n} \mathrm{multgeom}_A(\lambda_i) = n$

Ex 11.9 (Multiple choice and True/False questions)

a) i) Let A be a 3×3 matrix that has the eigenvalues -1, 1 and 2. Then

I) The rank of A is equal to

$(A) =$ $\quad (A)\ 1 \quad (B)\ 2 \quad (C)\ 0 \quad (D)\ 3$

II) The determinant of $A^T A$ is equal to

$(A)\ 2 \quad (B)\ 4 \quad (C)\ 0 \quad (D)\ 3$

III) The determinant of $A + I$ is equal to

$(A)\ 1 \quad (B)\ 6 \quad (C)\ 0 \quad (D)\ -1$

IV) The determinant of A^{-1} is equal to

$(A)\ -2 \quad (B)\ -1 \quad (C)\ 1 \quad (D)\ -1/2$

ii) Consider the matrices

$$A = \begin{pmatrix} 2 & 0 & 8 \\ 1 & 4 & -4 \\ 1 & 2 & -3 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 3 & -3 & 1 \end{pmatrix}.$$

Then the following matrices are diagonalizable:

(A) both A and B $\quad$ (B) only B $\quad$ (C) only A $\quad$ (D) neither A nor B.

iii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6x_1 + 2x_2 + 4x_3 \\ - 3x_2 + x_3 \\ 2x_1 + 8x_2 - x_3 \end{pmatrix}.$$

Let $\mathcal{E}$ and $\mathcal{F}$ be two bases of $\mathbb{R}^3$ given by

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \text{and} \qquad \mathcal{F} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}.$$

Then the matrix of T in the bases $\mathcal{E}$ (outgoing) and $\mathcal{F}$ (incoming) is

$$(A) \begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 0 \end{pmatrix} \quad (B) \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 5 & 0 \end{pmatrix} \quad (C) \begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 1 \\ 2 & 5 & 0 \end{pmatrix} \quad (D) \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

b) In the following, let A be an $n \times n$ matrix. Decide whether the following statements are always true or if they can be false.

i) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors of A, then they correspond to distinct eigenvalues.

ii) If A is invertible, then it is diagonalizable.

iii) If A is not invertible, then it is not diagonalizable.

iv) If A has fewer than n distinct eigenvalues, then A is not diagonalizable.

v) If $AP = PD$ for some diagonal matrix D, then all columns of P are eigenvectors of A.

vi) If $AP = PD$ for some diagonal matrix D, then A is diagonalizable.

vii) A has diagonalizable if A has n eigenvectors.

viii) If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.