MATH-111(en) FALL 2024
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SOLUTIONS for Homework 2

Ex 2.1 (The weekly linear system)

Solve the following linear system and write the solution in parametric form.

$$\begin{cases} x + 2y + 2t = 3 \\ y + z - 2t = 0 \\ x + 3y + z + t = 5 \\ 2x + 5y + z + 4t = 10 \end{cases}$$

Solution :
We find the augmented matrix and then apply row reduction :

$$\begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 3 & 1 & 1 & 5 \\ 2 & 5 & 1 & 4 & 10 \end{pmatrix} \xrightarrow[4)\to 4)-2\cdot 1)]{3)\to 3)-1)} \begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 & 4 \end{pmatrix} \xrightarrow[4)\to 4)-\cdot 2)]{3)\to 3)-2)} \begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{pmatrix}$$

$$\xrightarrow{4)\to 4)-2\cdot 3)} \begin{pmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[1)\to 1)-2\cdot 3)]{2)\to 2)+2\cdot 3)} \begin{pmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{1)\to 1)-2\cdot 2)} \begin{pmatrix} 1 & 0 & -2 & 0 & -9 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the reduced echelon form. We find that $t = 2$, $y = 4 - z$ and $x = -9 + 2z$. Hence we obtain

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -9+2z \\ 4-z \\ z \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 4 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

as the solution in parametric vector form.

Ex 2.2 (On $B(Ax) \neq A(Bx)$)

We first observe that : If $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$, then $Ax \in \mathbb{R}^m$ is well-defined. If $B \in \mathbb{R}^{p\times m}$, then the element $B(Ax) \in \mathbb{R}^p$ is well-defined.

a) Determine all values of m and p (depending on n) such that $A(Bx)$ is also defined.

b) For $n = 2$, consider the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Show that for $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ it holds that $B(Ax) \neq A(Bx)$. Find all vectors $x \in \mathbb{R}^2$ such that $B(Ax) = A(Bx)$.

Solution :
a) If $x \in \mathbb{R}^n$ and Bx is well-defined, we must have $n = m$. Moreover, if $y = Bx \in \mathbb{R}^p$ and Ay is well-defined, we must have that $p = n$. Hence $A(Bx)$ is defined if and only if $m = p = n$.

b) A direct computation based on the definition of the matrix-vector product yields that

$$Ax = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix}, \qquad Bx = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$$

and then

$$B(Ax) = B\begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}, \qquad A(Bx) = A\begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 0 \end{pmatrix}.$$

Thus these terms are equal if and only if

$$\begin{pmatrix} 2x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} \iff \begin{pmatrix} -x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is a linear system with augmented matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

so that $x_1 = 0$ and $x_2 = 0$. In particular, for $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the terms are not equal and we found that they only agree for $x = 0$.

Ex 2.3 (Conversion to parametric vector form)
Write all solutions to the linear systems from Problems 1, 2, and 3 from Homework 1 in parametric vector form.

Solution :
When the solution is unique, it suffices to write the solution vector. Otherwise we perform the method of course, writing the equations for the basic variables in a vector and separating terms.

1. (a)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{29}{7} \\ \frac{6}{7} \end{pmatrix}.$$

(b)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -46 \\ -46 \\ 95 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} \frac{-4}{5} \\ \frac{1}{5} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{5} \\ \frac{-2}{5} \\ 0 \end{pmatrix} \text{ where } s \in \mathbb{R}.$$

2. (a)

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{7}{5} \\ \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{pmatrix} \text{ where } s \in \mathbb{R}.$$

(b) No solution.

(c)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ where } s, t \in \mathbb{R}.$$

3. (a)

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ where } s, t \in \mathbb{R}.$$

(b)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Ex 2.4 (A system with a parameter)
For

$$A = \begin{pmatrix} 1 & -1 & -3 \\ 2 & 0 & 1 \\ 0 & 2 & 7 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ a \end{pmatrix},$$

find an a such that $A\mathbf{x} = \mathbf{b}$ is consistent, and write the solution in this case in parametric vector form.

Solution :
As usual, we apply row reduction to find the echelon form.

$$\left(\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 7 & a \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 2 & 7 & -6 \\ 0 & 2 & 7 & a \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 2 & 7 & -6 \\ 0 & 0 & 0 & a+6 \end{array}\right)$$

So this matrix equation has a solution if and only if $a = -6$. In this case :

$$\left(\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 2 & 7 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 1 & 7/2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 7/2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow z \text{ is free}, \quad x = -\frac{1}{2}z, \quad y = -3 - \frac{7}{2}z.$$

Plugging this in, we get that the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z \\ -3 - \frac{7}{2}z \\ z \end{pmatrix}$ or in parametric vector form :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} == \lambda \cdot \begin{pmatrix} -1/2 \\ -7/2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

Ex 2.5 (Consistency for all right-hand sides ?)
Let

$$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}.$$

Is the equation $\mathbf{Ax} = \mathbf{b}$ consistent for all choices of $\mathbf{b} \in \mathbb{R}^2$? Determine the set of $\mathbf{b} \in \mathbb{R}^2$ for which the equation $\mathbf{Ax} = \mathbf{b}$ is consistent.

Solution :
Let $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. The augmented matrix of the system is equivalent to the row echelon matrix

$$\begin{pmatrix} -3 & 1 & b_1 \\ 0 & 0 & 2b_1 + b_2 \end{pmatrix}$$

Thus the equation $\mathbf{Ax} = \mathbf{b}$ is consistent only if $2b_1 + b_2 = 0$, i.e., the set of $\mathbf{b}$ for which $\mathbf{Ax} = \mathbf{b}$ is consistent is

$$\left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; 2b_1 + b_2 = 0 \right\}.$$

(This is the line passing through the origin and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.) In particular, there exist $\mathbf{b} \in \mathbb{R}^2$ for which $\mathbf{Ax} = \mathbf{b}$ is inconsistent, for instance $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Ex 2.6 (Homogeneous and inhomogeneous systems)

Consider the two systems

$$\begin{cases} x_1 - 3x_2 - 2x_3 &= 0 \\ x_2 - x_3 &= 0 \\ -2x_1 + 3x_2 + 7x_3 &= 0 \end{cases} \qquad \begin{cases} x_1 - 3x_2 - 2x_3 &= -5 \\ x_2 - x_3 &= 4 \\ -2x_1 + 3x_2 + 7x_3 &= -2 \end{cases}.$$

For each of these systems, write the solution in its parametric vector form and give a geometric description of the solution space. For example, say what "shape" it is and whether or not it contains the origin.

Solution :
The reduced row echelon matrices corresponding to the systems are respectively

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|c} 1 & 0 & -5 & 7 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

In both cases, x_3 is a free variable and the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} + s \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

Recall that the solution space of a system of linear equations in three variables (in case it is not the empty set) always is a point, a line or a plane. It contains the origin if and only if the system is homogeneous.

The solution to the first system corresponds to a line passing through the origin. The solution of the second system corresponds to a line passing through the point $(7, 4, 0)$ and parallel to the first line.
Note that it actually suffices to consider the inhomogeneous system. The solution of the homogeneous system is given by the part involving the free variables.

Ex 2.7 (A homogeneous equation)

For the following matrix $\mathbf{A}$, write the solution of $\mathbf{Ax} = \mathbf{0}$ in its parametric vector form.

$$\mathbf{A} = \begin{pmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution :
The row echelon form of the matrix $\mathbf{A}$ shows that there are 3 basic variables (x_1, x_3 and x_6) and 3 free variables (x_2, x_4 and x_5), thus any solution may be written as a linear combination in the following way :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = s \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -8 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Ex 2.8 (On homogeneous systems)
Recall that a system $A\mathbf{x} = \mathbf{b}$ is *homogeneous* if $\mathbf{b} = \mathbf{0}$ (where $\mathbf{0}$ is the vector in which every entry is zero). Determine if the following statements are true or false. Justify your answer. (*Justify* means : if it is true explain/prove why ; if it is false, give a counterexample.)

1. If $\mathbf{x}$ is a solution to a homogeneous system, then $2\mathbf{x}$ is also a solution.
2. If $\mathbf{x}$ is a solution to an inhomogeneous system, then $2\mathbf{x}$ is also a solution.
3. If $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{0}$ and $\mathbf{y}$ is a solution of $A\mathbf{y} = \mathbf{b}$, then $\mathbf{v} = \mathbf{x} + \mathbf{y}$ is a solution of $A\mathbf{v} = \mathbf{b}$.
4. If $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{0}$ and $\mathbf{y}$ is a solution of $A\mathbf{y} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{v} = 2\mathbf{x}+3\mathbf{y}$ is a solution of $A\mathbf{v} = \mathbf{b}$.

Solution :

1. *If $\boldsymbol{x}$ is a solution to a homogeneous system, then $2\boldsymbol{x}$ is also a solution.*
   True : Write the system as $A\mathbf{x} = 0$. If $\mathbf{x}$ is a solution, so $A\mathbf{x} = 0$, then we also have $A(2\mathbf{x}) = 2 \cdot A\mathbf{x} = 2 \cdot 0 = 0$, so $2\mathbf{x}$ is a solution.
2. *If $\boldsymbol{x}$ is a solution to an inhomogeneous system, then $2\boldsymbol{x}$ is also a solution.*
   False : Write the system as $A\mathbf{x} = \mathbf{b}$. If $\mathbf{x}$ is a solution, so $A\mathbf{x} = \mathbf{b}$, then we have $A(2\mathbf{x}) = 2 \cdot A\mathbf{x} = 2\mathbf{b}$. Since the system is *in*homogeneous, we have $\mathbf{b} \neq 0$, so $A(2\mathbf{x}) = 2\mathbf{b}$ implies that $A(2\mathbf{x}) = \mathbf{b}$ does not hold, which means that $2\mathbf{x}$ is *not* a solution of $A\mathbf{x} = \mathbf{b}$.
3. *If $\boldsymbol{x}$ is a solution of $A\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{y}$ is a solution of $A\boldsymbol{y} = \boldsymbol{b}$, then $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$ is a solution of $A\boldsymbol{v} = \boldsymbol{b}$.*
   True : We have $A\mathbf{v} = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = 0 + \mathbf{b} = \mathbf{b}$.
4. *If $\boldsymbol{x}$ is a solution of $A\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{y}$ is a solution of $A\boldsymbol{y} = \boldsymbol{b}$ where $\boldsymbol{b} \neq \mathbf{0}$, then $\boldsymbol{v} = 2\boldsymbol{x}+3\boldsymbol{y}$ is a solution of $A\boldsymbol{v} = \boldsymbol{b}$.*
   False : We have $A\mathbf{v} = A(2\mathbf{x} + 3\mathbf{y}) = 2A\mathbf{x} + 3A\mathbf{y} = 2 \cdot 0 + 3 \cdot \mathbf{b} = 3 \cdot \mathbf{b} \neq \mathbf{b}$.

Ex 2.9 (Two related systems)

Solve the following two systems of linear equations and write the solutions in parametric vector form. What is the connection between the solution sets of the two systems ?

$$\begin{cases} x + y - 3z &= 0 \\ 3x + 7y - 13z &= 0 \\ x - y - z &= 0 \end{cases} \qquad \begin{cases} x + y - 3z &= 0 \\ 3x + 7y - 13z &= -4 \\ x - y - z &= 2 \end{cases}$$

Solution :

$$\begin{cases} x+y-3z &= 0 \\ 3x+7y-13z &= 0 \\ x-y-z &= 0 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 3 & 7 & -13 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow z \text{ free, } x = 2z, y = z$$

$$\text{parametric vector form : } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Note that to solve the second system we use the same row reduction steps, but now with a different last column.

$$\begin{cases} x+y-3z &= 0 \\ 3x+7y-13z &= -4 \\ x-y-z &= 2 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 3 & 7 & -13 & -4 \\ 1 & -1 & -1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 4 & -4 & -4 \\ 0 & -2 & 2 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow z \text{ free, } x = 2z+1, y = z-1$$

$$\text{parametric vector form : } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

The connection between the two solution sets is that the solutions of the second system are exactly the solutions of the first system plus the vector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. This has to be the case since the systems only differ by their right-hand sides, i.e, one is the homogeneous system, while the other one is a corresponding inhomogeneous system.

Ex 2.10 (Multiple choice and True/False questions)

(a) Let

$$A = \begin{pmatrix} 2 & 3 & -3 \\ -2 & 0 & 5 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}.$$

Then the solution $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of the matrix equation $A\mathbf{x} = \mathbf{b}$ is such that

(A) $x_1 = 3$ $\quad (B)$ $x_1 = 1$ $\quad (C)$ $x_1 = 4$ $\quad (D)$ $x_1 = -2$.

(b) The Linear System

$$\begin{cases} x - 7y + 2z - 13t = 5 \\ 2x - 4y + 2z - 12t = 3 \\ 3x + y + 2z - t = 1 \\ 3x + 2y + 2z + 4t = 0 \end{cases}$$

has

(A) no solution
(B) a unique solution
(C) a straight line as its solution set
(D) a plane as its solution set.

c) Decide whether the following statements are always true or if they can be false.
   (i) A vector $\mathbf{b}$ is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.
   (ii) If the augmented matrix $[A\ \mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.
   (iii) The solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the equation $A\mathbf{x} = \mathbf{0}$.
   (iv) The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

Solution :

(a) **The answer is (A).** By applying row operations one solves the linear system to find that $x_3 = 1$, $x_2 = -2$ and $x_1 = 3$.

(b) **The answer is (A).** By applying row operations one can show that the (reduced) echelon form of the augmented matrix has a row of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & b \end{pmatrix}$ with $b \neq 0$. Thus there exists no solution.

(c) (i) **True.** If the columns of A are $\mathbf{a}_1, \dots, \mathbf{a}_n$, then $A\mathbf{x}$ is equal to $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$. Therefore, solving $A\mathbf{x} = \mathbf{b}$ corresponds to finding the coefficients for a representation of $\mathbf{b}$ as a linear combination of the columns of A.
   (ii) **False.** It is possible that all rows of the augmented matrix have a pivot position without the rightmost column being a pivot column (which is the relevant condition for consistency). For example, the augmented matrix for the consistent equation $x_1 = 5$ is $[1\ 5]$, whose only row has a pivot position.
   (iii) **False.** This is true only if $\mathbf{p}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}$. In fact, $A\mathbf{v}_h = \mathbf{0}$ and $A\mathbf{w} = \mathbf{b}$ imply that $\mathbf{b} = A\mathbf{w} - A\mathbf{v}_h = A(\mathbf{w} - \mathbf{v}_h) = A\mathbf{p}$.
   (iv) **True.** As $A\mathbf{0} = \mathbf{0}$, $A\mathbf{0} = \mathbf{b}$ implies that $\mathbf{b} = \mathbf{0}$.

Ex 2.11 (Linear combinations and linear dependency : Proofs)

(a.) Finish the proof of Theorem 1.7 from class.
(b.) Prove Theorem 1.8 from class.
(c.) Prove Theorem 1.9 from class.

Tip : if you cannot come up with a proof from the top of your head : try to first state the assumption then the goal and write both out according to the definition. (See e.g. first part of proof of Theorem 1.7 from class.) Then try to work from one towards the other. If you still cannot do it, it may be fruitful to make up an example (choose a couple of vectors in $\mathbb{R}^3$ with actual numeric entries) that illustrates the given scenario. This might give you an idea of where to start the formal proof. Coming up with your own examples to explain something to yourself is a crucial part of becoming independent in your mathematical thinking process!

Solution :
Recall that the set $V := \{v_1, \dots, v_k\} \subset \mathbb{R}^n$ is linearly independent if the only solution $(\lambda_1, \dots, \lambda_k)$ for the equation $\sum_{i=1}^{k} \lambda_i v_i = 0$ is $\lambda_i = 0$ for all $i \in \{1, \dots, k\}$.

(a.) To complete the proof of Theorem 1.7, suppose there is a $j \in \{1, \ldots, k\}$ such that $v_j = \sum_{i \neq j} c_i v_i$, where $c_i \in \mathbb{R}$ are scalars. We consider two scenarios.
(1) If all c_i are 0, then v_j is the zero vector in $\mathbb{R}^n$ and hence the set V is linearly dependent (see Part (b.) below).
(2) If some c_i are nonzero, then we have that $\sum_{i=1}^{k} \lambda_i v_i = 0$ under the choice $\lambda_i = -c_i$ (for $i \neq j$) and $\lambda_j = 1$, that is,

$$\sum_{i=1}^{k} \lambda_i v_i = \lambda_1 v_1 + \cdots + v_j + \cdots + \lambda_k v_k = -c_1 v_1 + \cdots + \sum_{i \neq j} c_i v_i + \cdots - c_k v_k = 0.$$

Hence, the set V is also in this scenario linearly dependent.

(b.) (i) Let $\{v_1, ..., v_k\} \subset \mathbb{R}^n$ and $k > n$. We want to show that $\{v_1, ..., v_k\}$ are linearly dependent. Consider the matrix $A := (v_1, ..., v_k)$ (that is, the matrix that has $v_1, ..., v_n$ as its columns). Hence, by Theorems from class, linear dependence of $\{v_1, ..., v_k\}$ is equivalent to $Ax = 0$ having not only $x = 0$ (zero vector) as a solution.
The assumption $k > n$ means that A has more columns than rows. Hence, every echelon form $\tilde{A}$ of A will have free variables. Therefore, in case one solution exists (and this is indeed the case, because the zero-vector is already known to be a solution), then infinitely many solutions exists.

(ii) Let $\{v_1, ..., v_k\} \subset \mathbb{R}^n$ and $v_j = 0$. Then we choose scalars $\lambda_j = 0$ for all $i \in \{0, ..., j-1, j+1, ..., k\}$ and we choose $\lambda_j = 1$, Then $\lambda_1 v_1 + ... + \lambda_k v_k = 0$ but not all scalars λ_i are zero. Hence the $\{v_1, ..., v_k\}$ are dependent.

(c.) (i) If $V := \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ is linearly dependent there are some scalars λ_i, some being nonzero, such that $\sum_{i=1}^{k} \lambda_i v_i = 0$. Then, $\{V, v_{k+1}\}$ is again linearly dependent since $\sum_{i=1}^{k+1} \lambda_i v_i = \sum_{i=1}^{k} \lambda_i v_i + 0 \cdot v_{k+1} = 0$, that is, we can set $\lambda_{k+1} = 0$.

(ii) Suppose $V := \{v_1, \ldots, v_{k-1}\} \subset \mathbb{R}^n$ is not linearly independent, there must be some scalars λ_i, some nonzero, such that $\sum_{i=1}^{k-1} \lambda_i v_i = 0$. However, then, $\{V, v_k\}$ cannot be linearly independent since $\sum_{i=1}^{k} \lambda_i v_i = \sum_{i=1}^{k-1} \lambda_i v_i + 0 \cdot v_k = 0$.

Important to know : In proof-based exercises, usually many different solutions are possible and correct. In addition, it is often up to your personal taste, whether you use more formulas or rather explain most things in written text. The most important thing is, that your statements (in words of formulas) are unambiguous and that all logical steps are accounted for (no gaps in the argument).