Duration : 144 minutes

Linear Algebra Exam

Common part

Fall 2023

Questions

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

Notation (all standard)

– $\mathbb{R}$ denotes the set of real numbers.
– For a matrix A, $a_{ij} \in \mathbb{R}$ denotes the entry of A in row i and column j.
– For a vector $x \in \mathbb{R}^n$, x_i denotes the ith coordinate of x.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– $\mathbb{R}^{m\times n}$ is the vector space of $m \times n$ matrices.
– The scalar or inner product of vectors $x, y \in \mathbb{R}^n$ is defined as $x \cdot y = x^T y$.
– The length of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x \cdot x}$.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct response.

Question 1: The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

has a QR-decomposition such that

☐ $r_{12} = 0$ ☐ $r_{12} = \frac{1}{2}\sqrt{2}$ ☐ $r_{12} = \sqrt{2}$ ☐ $r_{12} = 1$

Question 2: Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \right\}$$

be two ordered bases of $\mathbb{R}^3$. Let P be the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$ (i.e., the matrix that satisfies $[x]_{\mathcal{C}} = P[x]_{\mathcal{B}}$ for all $x \in \mathbb{R}^3$). Then, the second row of P is

☐ $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}$

☐ $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}$

Question 3: Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 3 & 0 \\ 2 & \sqrt{3} & \pi & 3 & \sqrt{2} \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & \pi & 3 & \sqrt{2} \\ \sqrt{3} & 1 & \pi & 3 & \sqrt{2} \end{pmatrix}.$$

Then

☐ $\det(A) = 12\pi$ ☐ $\det(A) = -6\pi$ ☐ $\det(A) = 0$ ☐ $\det(A) = \sqrt{6}\pi$

Question 4: The line that best approximates (in the sense of least squares) the following points $(-3, -7), (-2, -3), (0, 3), (3, 7)$ is

☐ $y = \frac{8}{7} + \frac{16}{7}x$ ☐ $y = -\frac{8}{7} + \frac{16}{7}x$ ☐ $y = \frac{16}{7} + \frac{8}{7}x$ ☐ $y = -\frac{16}{7} + \frac{8}{7}x$

Question 5: Let $\mathcal{B} = \{2 - t,\ t + t^2,\ -1 + t^3,\ -1 - t + 2t^2\}$ be an ordered basis of $\mathbb{P}_3$. The fourth coordinate of the polynomial $p(t) = t + 2t^2 + 3t^3$ with respect to the basis $\mathcal{B}$ is

☐ 3 ☐ $-\frac{1}{7}$ ☐ -7 ☐ $\frac{1}{7}$

Question 6 : Let

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}.$$

The eigenvalues of A are

☐ -3 and 2 ☐ -2 and 3 ☐ -1 and 2 ☐ -1 and 1

Question 7 : Let

$$w_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}.$$

If b is the orthogonal projection of y onto $W = \text{Span}\{w_1, w_2, w_3\}$, then

☐ $b_3 = \frac{5}{4}$ ☐ $b_3 = 20$ ☐ $b_3 = \frac{7}{8}$ ☐ $b_3 = 14$

Question 8 : Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

If $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is a least-squares solution of the equation $Ax = b$, then the least-squares error of the approximation of b by Ax^* is

☐ $\|b - Ax^*\| = \sqrt{2}$ ☐ $\|b - Ax^*\| = 6$ ☐ $\|b - Ax^*\| = 0$ ☐ $\|b - Ax^*\| = \sqrt{6}$

Question 9 : The system of linear equations

$$\begin{cases} x_1 + 2x_2 + 5x_3 - 4x_4 = 0 \\ \quad\;\; x_2 + 2x_3 + x_4 = 7 \\ \quad\;\; x_2 + 3x_3 - 5x_4 = -4 \\ 2x_1 + 3x_2 + 4x_3 - 3x_4 = 1 \end{cases}$$

has a unique solution which satisfies

☐ $x_1 = 3$ ☐ $x_1 = 2$ ☐ $x_1 = -2$ ☐ $x_1 = -3$

Question 10 : Let

$$A = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 5 & -1 \\ 1 & -1 & 2 & 2 \\ 3 & 1 & 0 & 1 \end{pmatrix}.$$

If $B = A^{-1}$ is the inverse of the matrix A, then

☐ $b_{33} = \frac{4}{39}$ ☐ $b_{41} = \frac{1}{3}$ ☐ $b_{43} = \frac{2}{3}$ ☐ $b_{33} = -\frac{1}{13}$

Question 11: Let W be the vector space of $2{\times}2$ symmetric matrices and let $T\colon \mathbb{P}_2 \to W$ be the linear transformation defined by

$$T(a+bt+ct^2) = \begin{pmatrix} a & b-c \\ b-c & a+b+c \end{pmatrix} \qquad \text{for all } a,b,c \in \mathbb{R}.$$

Let

$$\mathcal{B} = \{1,\, 1-t,\, t+t^2\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be ordered bases of $\mathbb{P}_2$ and W, respectively. The matrix A associated to T relative to the bases $\mathcal{B}$ of $\mathbb{P}_2$ and $\mathcal{C}$ of W (i.e., the matrix satisfying $[T(p)]_{\mathcal{C}} = A[p]_{\mathcal{B}}$ for all $p \in \mathbb{P}_2$) is

- ☐ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Question 12: The matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

- ☐ is invertible but not diagonalizable
- ☐ is invertible and diagonalizable
- ☐ is neither invertible nor diagonalizable
- ☐ is diagonalizable but not invertible

Question 13: Let $A = \begin{pmatrix} 2 & 0 & 3 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$. Then

- ☐ all eigenvalues of A have the same geometric multiplicity
- ☐ $\lambda = 4$ is an eigenvalue of A of algebraic multiplicity 2
- ☐ all eigenvalues of A have the same algebraic multiplicity
- ☐ $\lambda = 2$ is an eigenvalue of A of geometric multiplicity 2

Question 14: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x-y \\ -5x+6y \\ x+y \end{pmatrix}.$$

Then

- ☐ T is injective but not surjective
- ☐ T is injective and surjective
- ☐ T is neither injective nor surjective
- ☐ T is surjective but not injective

Question 15 : Applying the Gram-Schmidt algorithm to the columns of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -2 \end{pmatrix}$$

yields an orthogonal basis of $\mathrm{Col}(A)$ given by the vectors

☐ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ -2 \end{pmatrix}$

☐ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

☐ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \\ -2 \end{pmatrix}$

☐ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 16 : Let W be a subspace of $\mathbb{R}^n$ and let u and v be two vectors in $\mathbb{R}^n$.

If $u \in W$, then the inner product between u and v is equal to the inner product between u and the orthogonal projection of v onto W.

☐ TRUE ☐ FALSE

Question 17 : Let $T\colon \mathbb{P}_6 \to \mathbb{R}^{3\times 2}$ be a linear transformation. Then there exist $p, q \in \mathbb{P}_6$ such that $p \neq q$ and $T(p) = T(q)$.

☐ TRUE ☐ FALSE

Question 18 : Let $\{b_1, \dots, b_m\}$ be a basis of $\mathbb{R}^m$. If $A \in \mathbb{R}^{m\times n}$ is such that the equation $Ax = b_k$ has at least one solution for all $k = 1, \dots, m$, then $\mathrm{Col}(A) = \mathbb{R}^m$.

☐ TRUE ☐ FALSE

Question 19 : If $u_1, \dots, u_k$ are k orthonormal vectors in $\mathbb{R}^n$, then the orthogonal complement of $\mathrm{Span}\{u_1, \dots, u_k\}$ is a subspace of $\mathbb{R}^n$ of dimension $n - k$.

☐ TRUE ☐ FALSE

Question 20 : If $A, B \in \mathbb{R}^{n\times n}$ are two invertible matrices such that $A+B$ is not the zero matrix, then $A+B$ is also invertible.

☐ TRUE ☐ FALSE

Question 21 : Let $A \in \mathbb{R}^{4\times 4}$ be a rank 3 matrix. If u, v, w are linearly independent vectors in $\mathbb{R}^4$, then Au, Av, Aw are linearly independent vectors in $\mathbb{R}^4$.

☐ TRUE ☐ FALSE

Question 22 : Let $A \in \mathbb{R}^{3\times 3}$ be a diagonalizable matrix with eigenvalues 2, 3, -5. Then it follows that

$$\det(A^3) = -27000.$$

☐ TRUE ☐ FALSE

Question 23 : Let V and W be two vector spaces and $T\colon V \to W$ a linear transformation.
If $\dim(\mathrm{Ker}\, T) = \dim(V)$, then $\mathrm{Ran}(T) = \{0_W\}$.

☐ TRUE ☐ FALSE

Question 24 : Let $A \in \mathbb{R}^{m\times n}$ where $m < n$. If the reduced echelon form of the matrix A has exactly k zero rows, then the set of solutions of the homogeneous system $Ax = 0$ is a subspace of $\mathbb{R}^n$ of dimension $n - k$.

☐ TRUE ☐ FALSE

Question 25 : Let $A \in \mathbb{R}^{n\times n}$ and let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by $T(x) = Ax$, for all $x \in \mathbb{R}^n$. If A is such that $A^5 = 0$, then T is surjective.

☐ TRUE ☐ FALSE

Question 26 : If $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, then

$$\det(A - A^T) = \det(A) - \det(A^T).$$

☐ TRUE ☐ FALSE

Question 27 : Let q be an arbitrary polynomial of degree 3. Then the set

$$\{p \in \mathbb{P}_3 : q(0) - p(0) = 0\}$$

is a subspace of $\mathbb{P}_3$.

☐ TRUE ☐ FALSE

Question 28 : Let W be the subspace of $\mathbb{P}_5$ spanned by $p_1, p_2, p_3, p_4 \in \mathbb{P}_5$. If $\dim(W) = 4$, then there exist two polynomials $p_5, p_6 \in \mathbb{P}_5$ such that the set $\mathcal{B} = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ is a basis of $\mathbb{P}_5$.

☐ TRUE ☐ FALSE

Third part: open questions

– Answer in the empty space below using a black or dark blue ballpen.
– Your answer should be carefully justified, and all the steps of your argument should be discussed in details.
– Leave the check-boxes empty, they are used for the grading.

Question 29 : *This question is worth 3 points.*

☐0 ☐1 ☐2 ☐3 *Do not write here*

Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be linearly independent vectors.

Let A be a diagonalizable matrix in $\mathbb{R}^{n\times n}$ so that the vectors $v_1, \ldots, v_n$ are eigenvectors of A for the eigenvalues $\alpha_1, \ldots, \alpha_n$ respectively.

Let B be a diagonalizable matrix in $\mathbb{R}^{n\times n}$ so that the vectors $v_1, \ldots, v_n$ are eigenvectors of B for the eigenvalues $\beta_1, \ldots, \beta_n$ respectively.

Prove that $A - B$ is diagonalizable and satisfies

$$\det(A - B) = (\alpha_1 - \beta_1) \cdots (\alpha_n - \beta_n) .$$

Question 30 : *This question is worth 3 points.*

☐0 ☐1 ☐2 ☐3 *Do not write here*

Let A be a symmetric matrix in $\mathbb{R}^{3\times 3}$ whose eigenvalues are

$$\lambda_1 = 2, \quad \lambda_2 = -2 \quad \text{and} \quad \lambda_3 = 4\,.$$

Let c be a real number and let

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} c \\ 2 \\ 0 \end{pmatrix}$$

be eigenvectors of A for the eigenvalues λ_1, λ_2 and λ_3 respectively.

Determine the value of c and find the matrix A.