Duration : 144 minutes

EPFL

Linear Algebra Exam

Common part

Fall 2022

Questions

For the **multiple choice** questions, we give

+3 points if your answer is correct,

0 points if you give no answer or more than one,

−1 if your answer is incorrect.

For the **true/false** questions, we give

+1 points if your answer is correct,

0 points if you give no answer or more than one,

−1 points if your answer is incorrect.

Notation (all standard)

– $\mathbb{R}$ denotes the set of real numbers.
– For a matrix A, $a_{ij} \in \mathbb{R}$ denotes the entry of A in row i and column j.
– For a vector $x \in \mathbb{R}^n$, x_i denotes the ith coordinate of x.
– I_m denotes the $m\times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– $\mathbb{R}^{m\times n}$ is the vector space of $m\times n$ matrices.
– The scalar or inner product of vectors $x, y \in \mathbb{R}^n$ is defined as $x \cdot y = x^T y$.
– The length of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x \cdot x}$.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct response.

Question 1 : Let

$$A = \begin{pmatrix} 3 & -5 \\ 5 & 3 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Then the least squares solution $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ of $Ax = b$ satisfies

☐ $x_1^* = \frac{6}{17}$ ☐ $x_1^* = \frac{12}{35}$ ☐ $x_1^* = -\frac{6}{17}$ ☐ $x_1^* = -\frac{12}{35}$

Question 2 : Let R be the reduced echelon form of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

Then

☐ $r_{14} = -4$ ☐ $r_{14} = -3$ ☐ $r_{14} = -6$ ☐ $r_{14} = 0$

Question 3 : The regression line that best approximates (in the sense of least squares) the points $(-2, -1), (0, 1), (2, -2), (4, 1)$ is

☐ $y = \frac{2}{5} + \frac{3}{20}\, x$ ☐ $y = -\frac{2}{5} - \frac{3}{20}\, x$ ☐ $y = \frac{2}{5} - \frac{3}{20}\, x$ ☐ $y = -\frac{2}{5} + \frac{3}{20}\, x$

Question 4 : The linear system

$$\begin{cases} x - 2y + 3z = 1 \\ 2x + y - 4z = a \\ x - z = 2 \end{cases}$$

has solutions if and only if

☐ $a = -\frac{2}{9}$ ☐ $a = \frac{2}{9}$ ☐ $a = -\frac{9}{2}$ ☐ $a = \frac{9}{2}$

Question 5 : Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

be two ordered bases of $\mathbb{R}^3$. Let $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ be the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$, i.e., the matrix such that $\big[x\big]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\big[x\big]_{\mathcal{B}}$ for all $x \in \mathbb{R}^3$. The third column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is given by

☐ $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ☐ $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ ☐ $\begin{pmatrix} -1/2 \\ -1/2 \\ 2 \end{pmatrix}$

Question 6 : Let $t \in \mathbb{R}$ be a parameter. The vectors

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} t \\ -9 \\ 8 \end{pmatrix}$$

are linearly dependent if and only if

☐ $t \neq 5$ ☐ $t = 5$ ☐ $t \neq -5$ ☐ $t = -5$

Question 7 : Let $T\colon \mathbb{P}_2 \to \mathbb{R}^{2\times 2}$ be the linear map defined by

$$T(p) = \begin{pmatrix} p(0) & p(1) \\ p(-1) & p(0) \end{pmatrix}, \qquad \text{for all } p \in \mathbb{P}_2.$$

Let

$$\mathcal{B} = \{1,\, t+t^2,\, t-t^2\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

be ordered bases of $\mathbb{P}_2$ and $\mathbb{R}^{2\times 2}$, respectively. The matrix A associated to T relative to the bases $\mathcal{B}$ of $\mathbb{P}_2$ and $\mathcal{C}$ of $\mathbb{R}^{2\times 2}$ such that $\big[T(p)\big]_{\mathcal{C}} = A\big[p\big]_{\mathcal{B}}$ for all $p \in \mathbb{P}_2$ is given by

☐ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix}$

☐ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Question 8 : Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} a & 3b & c \\ d+2a & 3e+6b & f+2c \\ g & 3h & k \end{pmatrix}$$

be two 3×3 matrices. If $\det(A) = 1$, then we have

☐ $\det(B) = 1$ ☐ $\det(B) = 2$ ☐ $\det(B) = 3$ ☐ $\det(B) = 6$

Question 9 : The inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 6 & 4 & 5 \\ 5 & 3 & 4 \end{pmatrix}$$

is given by

☐ $A^{-1} = \begin{pmatrix} -1 & -3 & 4 \\ -1 & 1 & -1 \\ 2 & 3 & -4 \end{pmatrix}$ ☐ $A^{-1} = \begin{pmatrix} -1 & -3 & 1 \\ -1 & 1 & 0 \\ 2 & 3 & -1 \end{pmatrix}$

☐ $A^{-1} = \begin{pmatrix} -1 & 3 & -4 \\ -1 & -1 & 1 \\ 2 & -3 & 4 \end{pmatrix}$ ☐ $A^{-1} = \begin{pmatrix} -1 & 3 & -1 \\ -1 & -1 & 0 \\ 2 & -3 & 1 \end{pmatrix}$

Question 10 : The Gram-Schmidt algorithm applied to the columns of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & -3 \end{pmatrix}$$

yields an orthogonal basis of $\mathrm{Col}(A)$ given by the vectors

☐ $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -5 \end{pmatrix}$

☐ $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \\ -4 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Question 11 : The orthogonal projection of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ on the subspace generated by the first two columns of the matrix A from Question 10 is the vector

☐ $\begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 8 \\ -4 \\ 0 \\ 12 \end{pmatrix}$ ☐ $\begin{pmatrix} 18 \\ 2 \\ 0 \\ 16 \end{pmatrix}$

Question 12 : The matrix A from Question 10 has a QR-decomposition such that

☐ $r_{23} = \frac{3}{\sqrt{6}}$ ☐ $r_{23} = -\frac{\sqrt{6}}{3}$ ☐ $r_{23} = -2$ ☐ $r_{23} = -\sqrt{6}$

Question 13 : Let $A = \begin{pmatrix} 3 & 0 & 1 \\ 5 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix}$. Then

☐ $\lambda = 4$ is an eigenvalue with geometric multiplicity 2

☐ $\lambda = 3$ is an eigenvalue with algebraic multiplicity 1

☐ $\lambda = 3$ is an eigenvalue with geometric multiplicity 2

☐ $\lambda = 4$ is an eigenvalue with geometric multiplicity 1

Question 14: Let $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ be the linear map defined by $T(x) = Ax$ for all $x \in \mathbb{R}^3$, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 5 \\ 2 & 1 & 7 \\ 0 & 2 & 6 \end{pmatrix}.$$

Then

- ☐ T is injective but not surjective
- ☐ T is neither injective nor surjective
- ☐ T is injective and surjective
- ☐ T is surjective but not injective

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 15 : Let V be a subspace of $\mathbb{R}^4$ and let w_1 and w_2 be two vectors in $\mathbb{R}^4$. If w_1 and w_2 are linearly independent, then the vectors $\mathrm{proj}_V(w_1)$ and $\mathrm{proj}_V(w_2)$ are linearly independent.

☐ TRUE ☐ FALSE

Question 16 : Let $\{v_1, \ldots, v_k\}$ be an orthogonal set of vectors in $\mathbb{R}^n$. If $v_0 \in \mathbb{R}^n$ is such that $\{v_0, v_1, \ldots, v_k\}$ is an orthogonal set, then $v_0 \in \mathrm{Span}\{v_1, \ldots, v_k\}^\perp$.

☐ TRUE ☐ FALSE

Question 17 : If $A \in \mathbb{R}^{m\times n}$, then it holds that

$$\dim(\mathrm{Col}\, A) + \dim(\mathrm{Col}\, A^T) + \dim(\mathrm{Ker}\, A) + \dim(\mathrm{Ker}\, A^T) = m + n\,.$$

☐ TRUE ☐ FALSE

Question 18 : Let A and P be two $n\times n$ matrices. If P^TAP is symmetric, then A is symmetric.

☐ TRUE ☐ FALSE

Question 19 : Let $W = \{A \in \mathbb{R}^{2\times 2} : A = A^T\}$. Then W is a three-dimensional subspace of $\mathbb{R}^{2\times 2}$.

☐ TRUE ☐ FALSE

Question 20 : Let $A \in \mathbb{R}^{n\times n}$ and R be its reduced echelon form. Then

$$\det(A) = \det(R).$$

☐ TRUE ☐ FALSE

Question 21 : Let $A, B \in \mathbb{R}^{n\times n}$ be two matrices. If A and B have the same characteristic polynomial, then A and B have the same eigenvalues and for each eigenvalue λ we have

$$\dim(\mathrm{Ker}(A - \lambda I_n)) = \dim(\mathrm{Ker}(B - \lambda I_n))\,.$$

☐ TRUE ☐ FALSE

Question 22 : Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear map. Let d be the dimension of the range of T. Then $d \le \dim(W)$ and $d \le \dim(V)$.

☐ TRUE ☐ FALSE

Question 23 : If $A \in \mathbb{R}^{m\times n}$ is a matrix whose columns form a basis for $\mathbb{R}^m$, then for all $b \in \mathbb{R}^m$ the linear system $Ax = b$ has a unique solution.

☐ TRUE ☐ FALSE

Question 24 : Let V be an n-dimensional vector space and let $\mathcal{F} = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in V. If every subset of $\mathcal{F}$ containing $n-1$ elements is linearly independent, then $\mathcal{F}$ is a basis for V.

☐ TRUE ☐ FALSE

Question 25 : If $A, B \in \mathbb{R}^{n\times n}$ are two invertible matrices, then $(A+B)^2$ is invertible.

☐ TRUE ☐ FALSE

Question 26 : The set $\{p \in \mathbb{P}_n : p(-t) = -p(t) \text{ for all } t \in \mathbb{R}\}$ is a subspace of $\mathbb{P}_n$.

☐ TRUE ☐ FALSE

Question 27 : If v and w are two vectors in $\mathbb{R}^3$, then the matrix

$$A = v\,v^T - w\,w^T$$

is diagonalizable.

☐ TRUE ☐ FALSE

Third part: open questions

– Answer in the empty space below using a black or dark blue ballpen.
– Your answer should be carefully justified, and all the steps of your argument should be discussed in details.
– Leave the check-boxes empty, they are used for the grading.

Question 28 : *This question is worth 3 points.*

☐0 ☐1 ☐2 ☐3 *Do not write here*

Let $A \in \mathbb{R}^{m\times n}$ be a matrix such that its reduced echelon form has exactly k zero rows. Determine the rank of A and the dimension of $\mathrm{Ker}(A)$ in terms of m, n and k.

Question 29 : *This question is worth 3 points.*

☐0 ☐1 ☐2 ☐3 *Do not write here*

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Let $v \in \mathbb{R}^n$ be an eigenvector of A and $W = \mathrm{Span}\{v\}$. Show that if $y \in W^{\perp}$, then $Ay \in W^{\perp}$.

Question 30 : *This question is worth 3 points.*

☐0 ☐1 ☐2 ☐3 *Do not write here*

Let $A \in \mathbb{R}^{n\times n}$ and let O be the $n\times n$ zero matrix.

Show that if A is diagonalizable and $A^2 = O$, then $A = O$.