Exemple de recherche de solutions au sens des moindres carrés

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$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \underbrace{}_{\vec{v}_1} \; \underbrace{}_{\vec{v}_2}$$

colonnes lin. indépendantes
⇒ on va trouver une unique sol. au sens des moindres carrés

$$\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \operatorname{Im}(A)$$

$\vec{x} \in \operatorname{Im}(A)$, alors il est de la forme $\begin{pmatrix} \alpha \\ \alpha+\beta \\ \beta \end{pmatrix} = \alpha\vec{v}_1 + \beta\vec{v}_2$

2 méthodes: $A^TA\vec{x} = A^T\vec{b}$ à résoudre

$A = QR$ puis $\underset{\sim\sim\sim\sim\sim\sim}{\hat{x} = R^{-1}Q^T\vec{b}}$

① $$A^TA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ (symétrique!)}$$

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{A^TA\vec{x} = A^T\vec{b}} : \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} 2x_1 + x_2 = 0 \\ x_1 + 2x_2 = 1 \end{cases} \quad \dots \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix} = \hat{x}$$

② $A = QR$ $\qquad$ $(\vec{v}_1, \vec{v}_2)$ base de $\operatorname{Im}(A)$

$\vec{u}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ et $\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\qquad$ $W_1 = \operatorname{span}\{\vec{v}_1\}$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{W_1}(\vec{v}_2) = \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\1\end{pmatrix} \rightsquigarrow \text{prenons } \vec{u}_2 = \begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$$\vec{q}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$$Q = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{2} & 1/\sqrt{6}\\ 0 & 2/\sqrt{6}\end{pmatrix}$$

$$R = Q^T A$$

$$= \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{2} & \sqrt{2}/2\\ 0 & 3/\sqrt{6}\end{pmatrix} \text{ triang. sup.}$$

$$\hat{x} = R^{-1} Q^T \vec{b}$$

$$\det(R) = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$R^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix}3/\sqrt{6} & -\sqrt{2}/2\\ 0 & \sqrt{2}\end{pmatrix}$$

$$\hat{x} = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{6}\\ 0 & \sqrt{2}/\sqrt{3}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1/3\\ 2/3\end{pmatrix}$$