Série 9

Keywords: Bases, coordinates with respect to a basis, basis changes

Reminder. Let V, W be two vector spaces and a linear map $T : V \longrightarrow W$.

- **The image** of T is the set

$$\mathrm{Im}(T) = \{T(\vec{v}) \text{ with } v \in V\} \subset W$$

- **The kernel** of T is the set

$$\mathrm{Ker}(T) = \{\vec{v} \in V \text{ such that } T(\vec{v}) = 0_W\} \subset V$$

Question 1 Prove the following propositions:

a) $\mathrm{Im}(T)$ is a subspace of W

b) $\mathrm{Ker}(T)$ is a subspace of V

Question 2 Find a basis for the kernel and the image of the following matrices

$$A = \begin{pmatrix} -1 & 3 \\ -2 & 6 \\ -4 & 12 \\ 3 & -9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 1 \end{pmatrix}.$$

Question 3

a) Are the following polynomials in $\mathbb{P}_3(\mathbb{R})$ linearly independent?

(i) $p_1(t) = 1 - t^2$, $p_2(t) = t^2$, $p_3(t) = t$

(ii) $p_1(t) = 1 + t + t^2$, $p_2(t) = t + t^2$, $p_3(t) = t^2$

b) Do the polynomials p_1, p_2, p_3 in (ii) form a basis of $\mathbb{P}_3$? If yes, show that they form a generating set. If not, add one or more polynomials to obtain a basis.

Question 4 Let $a, b, c \in \mathbb{R}$ and a set $\mathcal{F} = \{p, q, r, s\}$ consisting of the four polynomials

$$p(t) = t^2 + t + 1,\ q(t) = t^2 + 2t + a,\ r(t) = t^3 + b,\ s(t) = t + c.$$

Then

- ☐ $\mathcal{F}$ is dependent when $a + c - 1 \neq 0$
- ☐ $\mathcal{F}$ forms a basis of $\mathbb{P}_3$ for certain values of the parameters a, b, c
- ☐ $\mathcal{F}$ is always dependent
- ☐ $\mathcal{F}$ forms a basis of $\mathbb{P}_4$ for certain values of the parameters a, b, c

Question 5 Let $\mathrm{Tr}\colon \mathrm{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}$ be the "trace" function defined by

$$\mathrm{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d.$$

Among the following sets of matrices, which one forms a basis for Ker(Tr)?

- ☐ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- ☐ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.
- ☐ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
- ☐ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

Question 6

a) Let $W = \mathrm{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ where $\vec{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v_3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Find $\dim(W)$.

b) Find a subset $\mathcal{B}$ of $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ such that $\mathcal{B}$ is a basis of W.

c) Complete the set $\{\vec{v_1} + \vec{v_2}\} \subset W$ to obtain a basis for W.

Reminder. Let V be a vector space, with bases $\mathcal{B} = \{\vec{b}_1, \ldots \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1 \ldots \vec{c}_n\}$ of V.

- **The coordinates** of $\vec{v} \in V$ in the basis $\mathcal{B}$ are the coefficients of the unique linear combination of $\vec{v}$ in $\mathcal{B}$, that is

$$[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad \Longleftrightarrow \quad \vec{v} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n.$$

- **The change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$** is defined by

$$P_{\mathcal{CB}} = \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \dots & [\vec{b}_1]_{\mathcal{C}} \end{pmatrix} (= [\mathcal{B}]_{\mathcal{C}})$$

and satisfies the basis change formula

$$[\vec{v}]_{\mathcal{C}} = P_{\mathcal{CB}} [\vec{v}]_{\mathcal{B}}.$$

Question 7 Express the coordinates of the vectors $\vec{v}$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ and write the change of basis matrix $P_{\mathcal{CB}}$.

a) $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

b) $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$.

Question 8 Let $\mathcal{C} = \vec{c}_1, \vec{c}_2$ and $\mathcal{D} = \vec{d}_1, \vec{d}_2$ be bases of $\mathbb{R}^2$, where $\vec{c}_1$ and $\vec{c}_2$ are linearly independent and $\vec{d}_1 = 6\vec{c}_1 - 2\vec{c}_2$ and $\vec{d}_2 = 9\vec{c}_1 - 4\vec{c}_2$.

a) Calculate the change of basis matrix $P_{\mathcal{CD}}$ from $\mathcal{D}$ to $\mathcal{C}$.

b) Calculate the change of basis matrix $P_{\mathcal{DC}}$ from $\mathcal{C}$ to $\mathcal{D}$.

c) For $\vec{x} = -3\vec{c}_1 + 2\vec{c}_2$, find $[\vec{x}]\mathcal{C}$ and $[\vec{x}]\mathcal{D}$.

Question 9 Let $p(t) = 2t^2 + t - 3$ and $\mathcal{B} = \{1 + t, t + t^2, -2 + t + t^2\}$ be a basis of $\mathbb{P}_2$. Calculate $[p(t)]_{\mathcal{B}}$.

Question 10 Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ be a basis of $M_{2\times 2}$. Calculate $[A]_{\mathcal{B}}$.

Question 11

a) Calculate the following determinant:

$$\begin{vmatrix} 6 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

b) Calculate the following determinants :

$$\begin{vmatrix} a & b & a \\ b & a & b \\ a+b & a+b & a+b \end{vmatrix}, \quad \begin{vmatrix} a & b & 0 \\ a & a+b & c \\ a & b & a \end{vmatrix}.$$

c) Let $A = \begin{pmatrix} 4 & 3 & 0 & 1 \\ 2 & 1 & 4 & 0 \\ 4 & 18 & 17 & 23 \\ 49 & 1 & 72 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 18 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 0 \\ 3 & 4 & 1 & 18 \end{pmatrix}$. Compute $\det(AB)$.

Question 12 Answer the following questions:

a) How many pivots must a 7×5 matrix have for its columns to be linearly independent?

b) How many pivots must a 5×7 matrix have for its columns to be linearly independent?

c) How many pivots must a 5×7 matrix have for its columns to span $\mathbb{R}^5$?

d) How many pivots must a 5×7 matrix have for its columns to span $\mathbb{R}^7$?

Question 13 Let V and W be two vector spaces, $T : V \longrightarrow W$ a linear transformation, and $\{\vec{v}_1, \dots, \vec{v}_k\}$ a subset of V. Prove the following statements:

a) If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent (linked), then $\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$ is also linearly dependent (linked).

b) If T is injective and $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent (free), then the set $\{T(\vec{v}_1), \dots, T(\vec{v}_k)\}$ is also linearly independent (free).