Série 8

Keywords: vector spaces, vector subspaces, linear mappings between vector spaces

Reminder. Let V be a vector space equipped with addition $+$ and scalar multiplication $\cdot$. A subset $W \subset V$ is a **vector subspace of V** if W satisfies the following three properties:

(1) $W \neq \varnothing$

(2) $\vec{v} + \vec{w} \in W, \quad \forall \vec{v}, \vec{w} \in W$

(3) $\lambda \cdot \vec{v} \in W, \quad \forall \vec{v} \in W$ and $\lambda \in \mathbb{R}$

Question 1 Let V be a vector space equipped with addition $+$ and scalar multiplication $\cdot$. Prove that a subset $W \subset V$ is a vector subspace if and only if W satisfies the "simplified characterization":

(1') $0_V \in W$

(2') $\lambda \cdot \vec{v} + \vec{w} \in W, \quad \forall \vec{v}, \vec{w} \in W$ and $\lambda \in \mathbb{R}$.

Question 2 Which of the following sets are vector subspaces of $\mathbb{R}^n$?

a) A solid cube in $\mathbb{R}^3$, centered at the origin.

b) The diagonal $\Delta = \{(x, x, \cdots, x) \in \mathbb{R}^n\}$.

c) A subset that has 2143 elements.

d) The union of all coordinate axes.

e) The set of points with integer coordinates.

Question 3 Let $V = \mathbb{F}(\mathbb{R}, \mathbb{R})$ be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following sets are subspaces of V?

a) $V_1 = \{f \in V \mid f(0) = f(1)\}$.

b) $V_2 = \{f \in V \mid f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$.

c) $V_3 = \{f \in V \mid f \text{ is bijective}\}$.

Question 4 Let $\mathbb{P}_n$, be the vector space of polynomials with real coefficients of degree less than or equal to n. Which of the following subsets of $\mathbb{P}_n$ are vector subspaces?

a) The set $V_1 = \{p \in \mathbb{P}_n \mid p(1) = 0\}$.

b) The set V_2 of all polynomials of exact degree n.

c) The set $V_3 = \{p \in \mathbb{P}_n \mid p(0) = 0\}$.

Question 5 Let $M_{n\times n}(\mathbb{R})$, be the vector space of $n \times n$ matrices with real coefficients. Which of the following sets are vector subspaces of $M_{n\times n}(\mathbb{R})$?

a) The set of upper triangular matrices in $M_{2\times 2}(\mathbb{R})$, i.e., matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, b, c \in \mathbb{R}$.

b) The set of matrices of the form $\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$ with $a, b \in \mathbb{R}$.

c) The set of matrices with trace zero.
*Note: the **trace** of a square matrix is the sum of the entries on its diagonal.*

d) The set of matrices with determinant zero.

e) The set of matrices A such that $A^4 = -I_n$.

Question 6 Let V be a vector space, and let $\vec{v_1}, \vec{v_2}, \vec{v_3} \in V$. Describe explicitly the subspace $\mathrm{Span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$ generated by $\vec{v_1}, \vec{v_2}, \vec{v_3}$ in the following cases:

a) $V = \mathbb{R}^3$, $\vec{v_1} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$, $\vec{v_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

b) $V = \mathbb{P}_3$, $\vec{v_1} = t$, $\vec{v_2} = t^2$, $\vec{v_3} = t^3$.

Question 7 We work in $V = \mathbb{P}_3$.
Let $p_1(t) = 1 - t$, $p_2(t) = t^3$, $p_3(t) = t^2 - t + 1$.
Does the polynomial $q(t) = t^3 - 2t + 1$ belong to $\mathrm{Vect}(p_1, p_2, p_3)$?

Question 8 Let $n \in \mathbb{N}$ be an integer with $n \geq 1$. For each of the following functions, determine and justify whether it is a linear transformation. If so, determine its kernel and image.

a) The determinant function $\det : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$.

b) The trace function $\mathrm{Tr} : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$.

c) The derivative function $D : \mathbb{P}_n \to \mathbb{P}_n$ that maps $p \in \mathbb{P}_n$ to its derivative p'.

Question 9 Let $\overrightarrow{w} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 3 & -5/2 \\ -3 & -2 & 4 \\ 2 & 4 & -4 \end{pmatrix}$.
Determine whether $\overrightarrow{w}$ is in $\mathrm{Im}(A)$, in $\mathrm{Ker}(A)$, or in both.

Question 10

1) Let $T : \mathbb{R} \to M_{2\times 2}(\mathbb{R})$ be defined by $T(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Then

- ☐ $\mathrm{Im}(T) = \{0\}$
- ☐ $\mathrm{Im}(T) = M_{2\times 2}(\mathbb{R})$
- ☐ T is linear and injective
- ☐ T is not linear

2) Let $V = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ and $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Then

- ☐ $V = \mathrm{Vect}(\vec{v_1}, \vec{v_2})$
- ☐ $\vec{v_2}$ spans V.
- ☐ $\vec{v_1}$ and $\vec{v_2}$ do not span V
- ☐ $V = \mathrm{Vect}(\vec{v_1})$.

3) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (x - y, y - z)$. Then

- ☐ $\mathrm{Ker}(T) \neq \{0\}$ and $\mathrm{Im}(T) \neq \mathbb{R}^2$
- ☐ $\mathrm{Ker}(T) = \{0\}$ and $\mathrm{Im}(T) \neq \mathbb{R}^2$
- ☐ $\mathrm{Ker}(T) = \mathbb{R}^3$ and $\mathrm{Im}(T) = \mathbb{R}^2$
- ☐ $\mathrm{Ker}(T) = \{(x, x, \cdots, x) \in \mathbb{R}^n\}$ and $\mathrm{Im}(T) = \mathbb{R}^2$

Question 11 Let $V = \mathbb{F}(\mathbb{R}, \mathbb{R})$ be the vector space of real-valued functions of a real variable, and let $f \in V$. Determine which of the following statements is true.

- ☐ If f is the zero vector of V, then $f(t) = 0$ for all $t \in \mathbb{R}$.
- ☐ If there exists $n \in \mathbb{N}$ such that $f(t) = 0\ \forall\ t \geq n$, then f is the zero vector of V.
- ☐ If there exists $t \in \mathbb{R}$ such that $f(t) = 0$, then f is the zero vector of V.
- ☐ If $f(q) = 0$ for all $q \in \mathbb{Q}$, then f is the zero vector of V.