Série 6

Keywords: injective, surjective and bijective linear applications (maps)

Question 1
Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map. Determine the necessary condition that m and n must satisfy so that

a) T is surjective,

b) T is injective,

c) T is bijective.

Question 2 Indicate for each statement whether it is true or false and briefly justify your answer.

a) If $\{\vec{v_1}, \vec{v_2}\}$ is a linearly independent set in $\mathbb{R}^n$ and $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map, then $\{T(\vec{v_1}), T(\vec{v_2})\}$ is a linearly independent set in $\mathbb{R}^m$.

b) If $\{\vec{v_1}, \vec{v_2}\}$ is a linearly dependent set in $\mathbb{R}^n$ and $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map, then $\{T(\vec{v_1}), T(\vec{v_2})\}$ is a linearly dependent set in $\mathbb{R}^m$.

c) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map. If the vectors $\vec{v_1}, \ldots, \vec{v_k}$ span $\mathbb{R}^n$ and are such that $T(\vec{v_j}) = \vec{0}$ for all $j \in \{1, \ldots, k\}$, then $T(\vec{v}) = \vec{0}$ for all $\vec{v} \in \mathbb{R}^n$.

d) If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $T(\vec{0}) = \vec{0}$, then T is a linear map.

e) If $T(\lambda\vec{u} + \mu\vec{v}) = \lambda T(\vec{u}) + \mu T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$, then $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map.

Question 3
Find the matrices associated with each of the following linear transformations, defined by the images of the vectors of the canonical basis:

a) $T : \mathbb{R}^2 \to \mathbb{R}^3$,
$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\-1\\2\end{pmatrix} \text{ et } T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-5\\3\\6\end{pmatrix}.$$

b) $T : \mathbb{R}^3 \to \mathbb{R}^3$,
$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ et } T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

c) $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation around the Oz axis by an angle of $60°$ (in the counterclockwise direction).

Question 4 The linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ given by

$$T(x, y, z) = (x + 2y + 3z\ ,\ 2x - 5y + 7z\ ,\ 3x - y - 2z\ ,\ y + 3z)$$

- ☐ is neither surjective nor injective
- ☐ is surjective but not injective
- ☐ is bijective
- ☐ is injective but not surjective

Question 5 Determine if the applications below are linear. Compute the canonically associated matrix for each linear application and determine whether the linear applications are injective, surjective, or bijective.

a) $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 \\ x_1 \\ x_2 - x_1 \\ x_2 \end{pmatrix}$$

b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2x_2 \\ -3x_1 \end{pmatrix}$$

c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{x_1} \\ 5x_2 \end{pmatrix}$$

d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_3 \end{pmatrix}$$

e) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \sin(x_1) \\ \cos(x_2) \end{pmatrix}$$

f) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_3 \\ x_1 - x_2 - x_3 \\ x_1 + x_3 + x_4 \end{pmatrix}$$

Question 6 Compute the canonically associated matrix for each linear application and determine whether the linear applications are injective, surjective, or bijective.

a) $T : \mathbb{R}^2 \to \mathbb{R}^3$,
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 4x_1 + 3x_2 \\ x_1 \\ x_2 \end{pmatrix}$$

b) $T : \mathbb{R}^3 \to \mathbb{R}$,
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto x_1 + x_2 + x_3$$

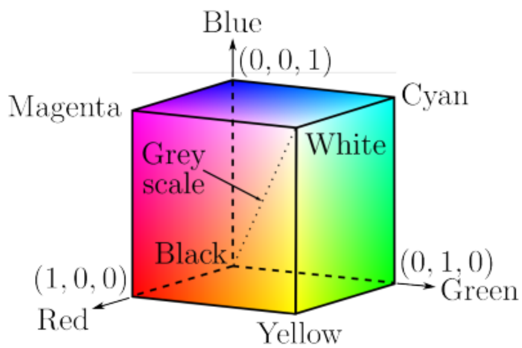
c) $T : \mathbb{R}^3 \to \mathbb{R}^3$,
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}$$

d) $T : \mathbb{R}^2 \to \mathbb{R}^2$,
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$$

e) $T : \mathbb{R}^2 \to \mathbb{R}^2$,
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

f) $T : \mathbb{R}^2 \to \mathbb{R}^2$,
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + x_2^2 \\ x_1 \end{pmatrix}$$

Question 7 Consider $\mathbf{C} \subset \mathbb{R}^3$, the RGB color cube[a] as shown below, with $R =$ Red, $G =$ Green, and $B =$ Blue.



a) Write the colors $C =$ Cyan, $Y =$ Yellow, and $M =$ Magenta as linear combinations of R, G, and B.

b) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation that transforms red into cyan, green into magenta, and blue into yellow. Write the matrix canonically associated with T.

c) Let $f : \mathbf{C} \longrightarrow \mathbf{C}$ be the function defined by $f(r, g, b) = (1 - r, 1 - g, 1 - b)$.

   i) Is $T(R) = f(R)$? And $T(G) = f(G)$? And $T(B) = f(B)$?

   ii) Are the functions T and f equal?

[a] It is the set $[0, 1] \times [0, 1] \times [0, 1]$; each point is a triplet (r, g, b) where $0 \leq r, g, b \leq 1$.