Série 5

Keywords:invertible matrice, linear transform, canonical matrix of a linear transformm

Question 1

a) Compute the inverse of the following matrix : $A = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$

(i) by using the general formula of a 2×2 matrix;

(ii) by writing the RREF of $(A \mid I_2)$.

b) Compute the inverse of $A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$ by writing the RREF of $(A \mid I_3)$.

Question 2

a) Is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ invertible ? If so, compute the inverse.

b) Find every solution of the homogeneous system $A\vec{x} = \vec{0}$.

c) Find every solution of the system $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Question 3 For which values of the parameters $a, b, c \in \mathbb{R}$ is the following matrix A invertible ?

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & c & 1 \end{pmatrix}$$

Whenever possible, give the inverse.

Question 4 Indicate for each statement whether it is true or false and briefly justify your answer.

a) Let A, B, and C be three matrices. Then $(AB)C = (AC)B$.

☐ False ☐ True

b) If A is an invertible matrix, then A^{-1} is also invertible.

☐ False ☐ True

c) The product of several invertible $n \times n$ matrices is not invertible.

☐ False ☐ True

d) If A is an invertible $n \times n$ matrix, then the equation $A\vec{x} = \vec{b}$ is consistent for any $\vec{b} \in \mathbb{R}^n$.

☐ False ☐ True

Question 5

a) The matrices are of size $n \times n$.

- ☐ Let A, B be two invertible matrices, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.
- ☐ Let A, B be two invertible matrices, then $A + B$ is invertible.
- ☐ There exists an invertible matrix A and a non-invertible matrix B such that AB is invertible.
- ☐ Let A, B be two matrices such that A or B is not invertible. Then AB is not invertible.

b) Let A be an $m \times n$ matrix and B an $n \times p$ matrix.

- ☐ If $m = n = p$, $A = A^T$, and $B = B^T$, then $(AB)^T = AB$.
- ☐ If $m = n$ and $A = A^T$, then A is diagonal.
- ☐ Then $(A^{-1})^T = (A^T)^{-1}$ if A is invertible.
- ☐ Then $(AB)^T = A^T B^T$.

c) Let A, B, C be three $n \times n$ matrices.

- ☐ If A is invertible and $AC = BC$, then $A = B$.
- ☐ If $C = C^T$ and $AC = BC$, then $A = B$.
- ☐ If C is invertible and $AC = BC$, then $A = B$.
- ☐ If $AC = BC$, then $A = B$.

Question 6

We consider the population of a region, divided into rural and urban populations. Let R_n and U_n denote the rural and urban populations in year n. Let a be the annual rate of rural exodus, and b the rate of urban exodus (both are assumed to be constant and given as percentages, so $0 \leq a, b \leq 1$).

a) Write the equations that give R_{n+1} and U_{n+1} as functions of R_n, U_n, a, and b.

b) Write these equations as a matrix equation $A \begin{pmatrix} R_n \\ U_n \end{pmatrix} = \begin{pmatrix} R_{n+1} \\ U_{n+1} \end{pmatrix}$ where A is a 2×2 matrix.

c) Take the values $a = 0.2$ and $b = 0.1$, as well as $R_0 = 100,000 = U_0$. Calculate the rural and urban populations in the third year.

d) Provide a formula for R_n and U_n in terms of R_0, U_0, and A^n.

Question 7 Find the associated canonical matrices of the following linear transformations :

a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

b) $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

c) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

Question 8 Consider $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_1 \end{pmatrix}$, and $T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto x_1 + x_2 + x_3$.

a) Write the associated canonocal matrices of T_1 and T_2 and write the product associated to the composition $T_2 \circ T_1$ such that $T_2 \circ T_1(\vec{x}) = T_2\left(T_1(\vec{x})\right)$ for all $\vec{x} \in \mathbb{R}^2$.

b) What is the domain of definition of $T_2 \circ T_1$? What is the codomain?

Question 9 Calculate the following matrix products, and indicate the corresponding compositions of linear transformations, with the dimensions of the spaces.
$T_{AB} : \mathbb{R}^{\cdots} \underset{T_{\cdots}}{\rightarrow} \mathbb{R}^{\cdots} \underset{T_{\cdots}}{\rightarrow} \mathbb{R}^{\cdots}$.

a) AB, with $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$.

b) ABC, with $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$.

c) ABC, with $A = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Question 10

a) In the plane, let S be the axial symmetry with axis $x = -y$. Describe its inverse if it exists. What are the matrices of these transformations?

b) Same question for H, the homothety with ratio 3.

c) Same question for R_θ, the rotation by angle θ centered at the origin.