Série 4

Keywords: matrix calculus, matrix product, power of a square matrix, transpose, inverse of a square matrix, elementary matrices.

Question 1 Consider

$$A = \begin{pmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Write the general solution of $A\vec{x} = 0_{\mathbb{R}^6}$.

Question 2 rite the solutions of the following systems $A\vec{x} = \vec{b}$ in the form $\vec{x} = \vec{p} + \vec{v}$, where $\vec{p}$ is a particular solution of the system, and $\vec{v}$ is the general solution of the homogeneous system $A\vec{x} = \vec{0}$.

a) $$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6 \end{cases}$$

b) $$\begin{cases} x_1 + x_2 - x_3 = 2 \\ 3x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 2x_2 - 2x_3 = 1 \end{cases}$$

Question 3 Find the values of a, b and c such that the following linear systems have solutions, and write those solutions.

$$\begin{cases} x - 2y + 3z + u = a \\ x + 3y - 2z + u = b \\ x - 7y + 8z + u = c \end{cases}$$

Question 4 Compute $A(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2)$, où

a)

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 0 \\ 1 & 3 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \alpha_1 = 2, \alpha_2 = 3;$$

b)

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}, \alpha_1 = -1, \alpha_2 = 1.$$

Question 5
Consider the matrices

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}, \ D = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ E = \begin{pmatrix} 1 & 4 \end{pmatrix}.$$

Compute the following products, if they exist.
If the products do not exist, explain why.

a) AB, BA, AC, CA, BC, CB, CD, EC, EA

b) AA^T, A^TA, BA^T, BC^T, C^TA, BD^T, D^TB

Question 6

a) Consider

$$A = \begin{pmatrix} 3 & -4 \\ -5 & 1 \end{pmatrix} \quad \text{et} \quad B = \begin{pmatrix} 7 & 4 \\ 5 & k \end{pmatrix}.$$

Find the values of $k \in \mathbb{R}$ such that $AB = BA$.

b) Consider

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}, \quad N = \begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix} \quad \text{et } T = \begin{pmatrix} 5 & 2 \\ 1 & -2 \end{pmatrix}.$$

Check that $MN = MT$, despite the fact that $N \neq T$.

Question 7
Indicate for each statement whether it is true or false and briefly justify your answer.

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a) If A and B are two 2×2 matrices whose columns are denoted by $\vec{a_1}, \vec{a_2}$ and $\vec{b_1}, \vec{b_2}$, then $AB = \begin{pmatrix} \vec{a_1} \cdot \vec{b_1} & \vec{a_2} \cdot \vec{b_2} \end{pmatrix}$. ☐ ☐

b) Let A, B, and C be three 3×3 matrices. Then $AB + AC = (B + C)A$. ☐ ☐

c) Let A and B be two $n \times n$ matrices. Then $A^T + B^T = (A + B)^T$. ☐ ☐

d) The transpose of a product of matrices is equal to the product of their transposes in the same order. ☐ ☐

Question 8

a) Determine the following 3×3 elementary matrices:

- E_1, which swaps the second and third rows;
- E_2, which multiplies the second row by 8;
- E_3, which adds 7 times the first row to the third.

b) Are the matrices E_1, E_2, and E_3 invertible? Why? If so, give their inverses and the inverse of the product $E_1\, E_2\, E_3$.

c) What elementary operation does each of the following matrices correspond to?

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; E_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Question 9 We consider elementary matrices of size 4×4.

a) Give the elementary matrix that swaps rows 2 and 4.

b) Give the elementary matrix that adds five times row 1 to row 3.

c) Give the elementary matrix that multiplies row 3 by 17.

d) Give the inverses of the matrices found in questions a, b, and c.

Question 10 Find which of the following matrices are invertible. Use the least amount of computation and justify your answer. You dont need to compute the inverse!

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -2 & -1 \\ -2 & -6 & 3 & 2 \\ 3 & 5 & 8 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 8 & 0 \\ 4 & 7 & 9 & 10 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 3 & -5 \\ 0 & 2 & -3 \\ 0 & -4 & 7 \\ -1 & 5 & -8 \end{pmatrix}.$$