Série 3

Keywords: Space $\mathbb{R}^n$, vector equations, linear combinations, linearly (in)dependent sets of vectors.

Question 1

Let's consider the vectors $\vec{a_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\vec{a_2} = \begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} \alpha \\ -5 \\ -3 \end{pmatrix}$. For which value(s) of α is the vector $\vec{b}$ a linear combination of $\vec{a_1}$ and $\vec{a_2}$?

- ☐ $\alpha = -\frac{5}{2}$
- ☐ $\alpha = -\frac{7}{2}$
- ☐ $\alpha = -\frac{3}{2}$ and $\alpha = -\frac{7}{2}$
- ☐ $\alpha = -\frac{3}{2}$ and $\alpha = -\frac{5}{2}$

Question 2

a) Consider the vectors $\overrightarrow{v_1} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$, $\quad \overrightarrow{v_2} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$, $\quad \overrightarrow{w} = \begin{pmatrix} 3 \\ 10 \\ h \end{pmatrix}$.

i) For which value(s) of h can the vector $\overrightarrow{w}$ be obtained as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$?

ii) In this case, what are the coefficients a_1, a_2 of the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$?

b) Consider the vector $\overrightarrow{v} = \begin{pmatrix} -5 \\ -3 \\ -6 \end{pmatrix}$, and the matrix $A = \begin{pmatrix} 3 & 5 \\ 1 & 1 \\ -2 & -8 \end{pmatrix}$. Is $\overrightarrow{v}$ in the plane of $\mathbb{R}^3$ spanned by the columns of A ? Justify your answer.

Question 3

Indicate for each statement whether it is true or false and briefly justify your answer.

1) The homogeneous system of linear equations represented by the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 7 \end{pmatrix}$ is consistent.

   ☐ True ☐ False

2) The inhomogeneous system of linear equations represented by the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 7 \end{pmatrix}$ is consistent.

   ☐ False ☐ True

3) If the coefficient matrix of a system of four equations in four unknowns has a pivot in each column, then the system is consistent.

   ☐ False ☐ True

4) If the augmented matrix of a system of four equations in four unknowns has a pivot in each row, then the system is consistent.

   ☐ False ☐ True

5) If $\vec{x}$ is a non-zero solution of $A\vec{x} = \vec{0}$, then none of the components of $\vec{x}$ are zero.

   ☐ False ☐ True

6) If A is an $m \times n$ matrix and $\vec{v}, \vec{w} \in \mathbb{R}^n$ are such that $A\vec{v} = \vec{0} = A\vec{w}$, then $A(\lambda\vec{v} + \mu\vec{w}) = \vec{0}$ for all $\lambda, \mu \in \mathbb{R}$.

   ☐ True ☐ False

Question 4 Are the following sets of vectors independant ? Do they span $\mathbb{R}^3$ (questions a) and b)) or $\mathbb{R}^2$ (question c)) ?

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

c) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

Question 5 Consider the vectors

$$\overrightarrow{v_1} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \quad \overrightarrow{v_3} = \begin{pmatrix} 1 \\ 2 \\ h \end{pmatrix}$$

- ☐ For all $h \in \mathbb{R}$, the vector $\overrightarrow{v_2}$ is linearly dependent on the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$.
- ☐ The vector $\overrightarrow{v_3}$ is linearly dependent on the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ for $h = 2$.
- ☐ The set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is linearly independent for $h = -2$.
- ☐ The set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is linearly independent for $h \neq -2$.

Question 6 Let $h \in \mathbb{R}$ and consider the vectors

$$\vec{v_1} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 7 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 1 \\ 2 \\ -10 \\ 1 \end{pmatrix} \quad \text{et} \quad \vec{v_3} = \begin{pmatrix} h+7 \\ 8 \\ 2h+1 \\ 25 \end{pmatrix}.$$

Thus $\vec{v_3}$ is a linear combinaison of $\vec{v_1}$ and $\vec{v_2}$ when

☐ $h = 4$ ☐ $h = 2$ ☐ $h = 1$ ☐ $h = -2$

Question 7 Let $b \in \mathbb{R}$. Thus the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ b \\ 0 \end{pmatrix}$$

are linearly independent if

☐ $b = 2$ ☐ $b = 4$ ☐ $b = 1$ ☐ $b = 3$

Question 8
Consider the linear system:

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6 \end{cases}$$

i) Write the system in matrix form $A\vec{x} = \vec{b}$.

ii) Write the system as a linear combination of the columns of the matrix A.

iii) Find the set of solutions to the equation $A\vec{x} = \vec{b}$.