Série 13

Keywords: Gram-Schmidt Algorithm, Least Squares Method, Orthogonal Matrices, Symmetric Matrices.

Reminder:

- Let a family of vectors $\{\vec{b}_1, \ldots, \vec{b}_k\} \subset \mathbb{R}^n$. Then, we can construct an orthogonal family $\{\vec{c}_1, \ldots, \vec{c}_k\} \subset \mathbb{R}^n$ as follows:
  - $\vec{c}_1 \overset{\text{def}}{=} \vec{b}_1, \qquad W_1 \overset{\text{def}}{=} \text{Span}\{\vec{c}_1\},$
  - $\vec{c}_2 \overset{\text{def}}{=} \vec{b}_2 - \text{Proj}_{W_1}(\vec{b}_2), \qquad W_2 \overset{\text{def}}{=} \text{Span}\{\vec{c}_1, \vec{c}_2\},$
  - $\vec{c}_3 \overset{\text{def}}{=} \vec{b}_3 - \text{Proj}_{W_2}(\vec{b}_3), \qquad W_3 \overset{\text{def}}{=} \text{Span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3\},$

    $\vdots$
  - $\vec{c}_i \overset{\text{def}}{=} \vec{b}_i - \text{Proj}_{W_{i-1}}(\vec{b}_i), \qquad W_i \overset{\text{def}}{=} \text{Span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_i\},$

    $\vdots$
  - $\vec{c}_k \overset{\text{def}}{=} \vec{b}_k - \text{Proj}_{W_{k-1}}(\vec{b}_k).$

- The **normal equation** of an inconsistent linear system $A\vec{x} = \vec{b}$ is the system
$$A^T A\hat{x} = A^T\vec{b}.$$
Its solution minimizes $\|A\vec{x} - \vec{b}\|$.

Question 1 Apply the Gram-Schmidt method to orthogonalize the bases of the following vector subspaces $W \subseteq \mathbb{R}^n$.

a) $W = \text{Span}\{w_1, w_2\}$, with $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

b) $W = \text{Span}\{w_1, w_2, w_3\}$, with $w_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $w_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

c) Provide an orthonormal basis for (a) and (b).

Question 2 Compute the least-squares solution of the linear system $A\vec{x} = \vec{b}$.

a) $A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$,

b) $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$,

c) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 5 \\ 6 \\ 6 \end{pmatrix}$;

Question 3 Consider the points

x_i	2	5	6	8
y_i	1	2	3	3

Compute the regression line that approximates the best these points.

Reminder:

- A matrix $Q \in M_{n\times n}$ is **orthogonal** if $Q^T Q = I_n$.
- A matrix $A \in M_{n\times n}$ is **symmetric** if $A^T = A$.
- **Spectral Theorem:** $A \in M_{n\times n}$ is symmetric if and only if it is **diagonalizable in an orthonormal basis**, i.e., we can write

$$A = PDP^T, \quad \text{where } D \in M_{n\times n} \text{ is diagonal, } P \in M_{n\times n} \text{ is orthogonal.}$$

Question 4

a) Show that if Q is an orthogonal matrix, then Q^T is also an orthogonal matrix.

b) Show that if U, V are $n\times n$ orthogonal matrices, then UV is also an orthogonal matrix.

c) Show that if Q is an orthogonal matrix and $\vec{x} \in \mathbb{R}^n$, then $||Q\vec{x}|| = ||\vec{x}||$.

d) Show that any real eigenvalue λ of an orthogonal matrix Q satisfies $\lambda = \pm 1$.

e) Let Q be an orthogonal matrix of size $n\times n$. Let $\{u_1, \dots, u_n\}$ be an orthogonal basis of $\mathbb{R}^n$. Show that $\{Qu_1, \dots, Qu_n\}$ is also an orthogonal basis of $\mathbb{R}^n$.

Question 5 Let A be a symmetric matrix of size $n \times n$.

a) Show that $(Av) \cdot u = v \cdot (Au)$ for all $u, v \in \mathbb{R}^n$.

b) Provide a 2×2 matrix B such that $(Bv) \cdot u \neq v \cdot (Bu)$ for some $u, v \in \mathbb{R}^2$.

c) Show that if A is invertible, then the inverse of A is also symmetric.

Question 6 Let $A \in M_{n \times n}$.

a) If $A^2 = 3I_n$, determine the only possible real eigenvalues of A.

b) If $A^2 + 2A + I_n = 0$, show that -1 is an eigenvalue of A.

Question 7 Diagonalize the following matrices in the form $P^TAP = D$, where P is an orthogonal matrix.

a) $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Question 8 The following data describe the potential in an electrical cable as a function of the cable's temperature.

i	T_i $[^\circ C]$	U_i $[V]$
1	0	-2
2	5	-1
3	10	0
4	15	1
5	20	2
6	25	4

We assume the potential follows the law $U = a + bT + cT^2$. Calculate a, b, c using the method of least squares.