Série 12

Keywords: Scalar product, norm, orthogonality, orthogonal complement, orthogonal projection.

Reminder: Let $\vec{v} = (v_1, \dots, v_n), \vec{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ be a subspace.

- The **scalar product** of $\vec{v}$ and $\vec{w}$ is:
$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n.$$
- The **norm** of $\vec{v}$ is:
$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$
- $\vec{v}$ and $\vec{w}$ are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.
- The **orthogonal complement of** W is the set:
$$W^\perp = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0,\ \forall \vec{w} \in W\}.$$
- If $A \in M_{m \times n}$, then $\mathrm{Im}(A)^\perp = \mathrm{Ker}(A^T)$.

Question 1 Prove that if W is a subspace of $\mathbb{R}^n$, then $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Question 2

a) Let $\vec{u} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix}$. Compute:
$$\vec{u} \cdot \vec{v}, \quad \vec{v} \cdot \vec{w}, \quad \frac{\vec{u} \cdot \vec{w}}{\|\vec{v}\|}, \quad \frac{1}{\vec{w} \cdot \vec{w}} \vec{w}, \quad \frac{\vec{u} \cdot \vec{w}}{\|\vec{v}\|} \vec{v}.$$

b) Compute the distance between $\vec{u}$ and $\vec{v}$, and between $\vec{u}$ and $\vec{w}$.

c) Compute the unit vectors corresponding to $\vec{u}$, $\vec{v}$, and $\vec{w}$ (pointing in the same direction as the original vector).

Question 3 Let $u, v \in \mathbb{R}^n$ such that $\|u\| = 2$, $\|v\| = 3$, and $u \cdot v = 5$.

a) Compute the scalar product of $u - 2v$ and $3u + v$.

b) Compute the scalar product of $au + bv$ and $bu + av$, where $a, b \in \mathbb{R}$.

c) Find all values of a and b such that the vectors $au + bv$ and $bu + av$ are orthogonal.

Question 4 For the following subspaces, provide a basis for the orthogonal complement:

$$W_1 = \operatorname{Span}\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}\right\}, \quad W_2 = \operatorname{Span}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right\},$$

$$W_3 = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \quad W_4 = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-2\\1\end{pmatrix}\right\}.$$

Question 5 Let $A = \begin{pmatrix}1 & -1 & 1\\ -2 & 2 & 3\\ -1 & 1 & 4\end{pmatrix}$ and $U = \ker(A)$. Then $U^\perp$ is equal to:

☐ $\operatorname{Im}(A^T)$ ☐ $\ker(A)$ ☐ $\operatorname{Im}(A)$ ☐ $\mathbb{R}^3$

Reminder:

- A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is **orthogonal** if:

$$\vec{v}_i \cdot \vec{v}_j = 0,\ \forall i \neq j.$$

  It is **orthonormal** if, in addition:

$$\|\vec{v}_i\| = 1,\ \forall i.$$

- The **orthogonal projection** of $\vec{v} \in \mathbb{R}^n$ onto a subspace $W \subset \mathbb{R}^n$ is the vector $\operatorname{proj}_W(\vec{v}) \in \mathbb{R}^n$ satisfying:

$$\operatorname{proj}_W(\vec{v}) \in W \quad \text{and} \quad \vec{v} - \operatorname{proj}_W(\vec{v}) \in W^\perp.$$

- If $\{\vec{u}_1, \dots, \vec{u}_k\}$ is an orthogonal basis for W, the projection is given by:

$$\mathrm{proj}_W(\vec{v}) = \frac{\vec{v} \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 + \cdots + \frac{\vec{v} \cdot \vec{u}_k}{\|\vec{u}_k\|^2}\vec{u}_k.$$

Question 6 Consider $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \in \mathbb{R}^3$.

a) u_1 and u_2 are orthogonal.

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b) Let $W = \mathrm{Span}\{u_1, u_2\}$. Compute $\mathrm{proj}_W(v)$.

☐ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 7/4 \\ 1/4 \\ 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 7/2 \\ 1/2 \\ 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$

c) Compute $v - \mathrm{proj}_W(v)$.

☐ $\begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$ ☐ $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ ☐ $\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$

Question 7 Consider $v = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $w_1 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Compute $\mathrm{proj}_W(v)$, avec $W = \mathrm{Span}\{w_1, w_2\}$.

☐ $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ ☐ $\begin{pmatrix} -4/3 \\ -1/3 \\ -5/3 \end{pmatrix}$ ☐ $\begin{pmatrix} 2/3 \\ 1/3 \\ 5/3 \end{pmatrix}$ ☐ $\begin{pmatrix} 4/3 \\ 1/3 \\ 5/3 \end{pmatrix}$

Question 8 Consider the vectors of $\mathbb{R}^4$

$$v = \begin{pmatrix} 2 \\ 4 \\ 0 \\ -1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ -3 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 5 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

et $W = \mathrm{Span}\{w_1, w_2\}$.

a) Compute $\mathrm{proj}_W(v)$

b) Compute the distance between v and W.

Question 9 Consider $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ two orthonormal bases of $\mathbb{R}^n$. Consider the matrices

$$U = \begin{pmatrix} u_1 & \ldots & u_n \end{pmatrix}, \ V = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix} \in M_{n\times n}.$$

Show that $U^T U = I_n$, $V^T V = I_n$ and UV is invertible.

Note : A matrix $U \in M_{n\times n}$ is **orthogonal** if $U^T U = I_n$.

Question 10 Consider $A = \begin{pmatrix} \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \end{pmatrix}$. Therefore

- ☐ A is not invertible
- ☐ $A^T A = I_3$
- ☐ $\frac{A}{\sqrt{6}}$ is orthogonal
- ☐ A is orthogonal

Question 11 State for each statement whether it is true or false, and briefly justify your answer.

a) A basis of a vector subspace $W \subset \mathbb{R}^n$ that consists of orthogonal vectors is an orthonormal basis.

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b) An orthogonal set $S = \{v_1, v_2, \ldots, v_p\}$ of nonzero vectors in $\mathbb{R}^n$ is linearly independent and therefore forms a basis of the subspace it spans.

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c) An orthonormal basis is an orthogonal basis, but the converse is generally false.

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d) If x does not belong to the vector subspace W, then $x - \mathrm{proj}_W(x)$ is nonzero.

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e) Any orthonormal set in $\mathbb{R}^n$ is linearly dependent.

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f) Let W be a vector subspace of $\mathbb{R}^n$. If $v \in W$ and $v \in W^\perp$, then $v = 0$.

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g) For a matrix A with dimensions $m \times n$, the product $A^T A$ is computed using the dot products of the columns of A. Therefore, if the columns of U are orthonormal, then $U^T U = I_n$.

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Question 12 For any $\vec{u}, \vec{v} \in \mathbb{R}^n$, show the following statements :

a) If $\{\vec{u}, \vec{v}\}$ are orthonormal, then $\|\vec{u} - \vec{v}\| = \sqrt{2}$.

b) $\vec{u} \cdot \vec{v} = \frac{1}{4}\left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2\right)$

c) $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$

d) Pythagoras's theorem : if $\vec{u}, \vec{v}$ are orthogonal, then

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$