Série 11

Keywords: Eigenvalues, eigenvectors, eigenspaces, diagonalization.

Reminder: Let $A \in M_{n\times n}$.

- A scalar $\lambda \in \mathbb{R}$ is an **eigenvalue of A**, and a vector $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, is an **eigenvector associated with** λ if
$$A\vec{v} = \lambda\vec{v}.$$
- The eigenvalues are the roots of the **characteristic polynomial**
$$P_A(\lambda) = \det(A - \lambda I_n).$$
- The **eigenspace** associated with an eigenvalue λ is the vector subspace
$$E_\lambda = \mathrm{Ker}(A - \lambda I_n) = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}.$$
- A is **diagonalizable** if and only if it admits n linearly independent eigenvectors. In this case, we can write
$$A = PDP^{-1} \quad \text{with } P = \begin{pmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$
where λ_i is an eigenvalue and $\vec{v}_i$ is an eigenvector associated with it.

Question 1 Let $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. Compute the eigenvalues and eigenspaces of A.

Question 2

Consider the matrix $A = \begin{pmatrix} -15 & 1 & -9 \\ 0 & 6 & 0 \\ 4 & 1 & 3 \end{pmatrix}$.

a) Is $\lambda = 6$ an eigenvalue of A?

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b) Are $\lambda = 1$ and $\lambda = -9$ eigenvalues of A?

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Question 3

Let A be the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$. Show that 0 is an eigenvalue of A and calculate the associated eigenspace.

Question 4

Let A be an invertible 3×3 matrix, and let λ be an eigenvalue of A.

- ☐ Then λ is an eigenvalue of A^{-1}.
- ☐ Then λ is an eigenvalue of $-A$.
- ☐ Then λ^{-1} is an eigenvalue of A^{-1}.
- ☐ Then λ^{-1} is an eigenvalue of $-A$.

Question 5

Let A be a 2×2 matrix that is not invertible. Then:

- ☐ Every vector in $\mathbb{R}^2$ is an eigenvector of A.
- ☐ A has no real eigenvalues.
- ☐ 0 is an eigenvalue of A.
- ☐ A is the zero matrix.

Question 6

Let A be an $n \times n$ matrix and $k \geq 2$ an integer. Verify that if λ is an eigenvalue of A with eigenvector $\vec{v}$, then λ^k is an eigenvalue of

$$A^k = \underbrace{A\,A\,\cdots\,A}_{k \text{ times}}$$

with eigenvector $\vec{v}$.

Question 7
Let $n \geq 2$ and $k \geq 2$ be integers.

a) There exists a diagonal matrix that is not diagonalizable.

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b) The $n \times n$ zero matrix is diagonalizable.

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c) Every upper triangular matrix is diagonalizable.

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d) If A is a diagonalizable $n \times n$ matrix, then A^k is diagonalizable.

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e) If A is an $n \times n$ matrix and A^k is diagonalizable, then A is diagonalizable.

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Question 8
Let

$$A = \begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Show that $A = PDP^{-1}$, and deduce the eigenvalues and eigenspaces of A.

b) Provide an expression for A^n.

Question 9 Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

a) Compute the eigenvalues and eigenspaces of A.

b) Provide an expression for A^k.

Question 10
For the following matrices, calculate the eigenvalues and eigenspaces, and determine which ones are diagonalizable:

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix},$$

$$C = \begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \; D = \begin{pmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Question 11 Let A be an $n \times n$ matrix. Indicate whether the following statements are true or false (justify your answers):

a) A is diagonalizable if and only if it has n distinct eigenvalues.

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b) If A is diagonalizable, then A is invertible.

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c) If A is invertible, then A is diagonalizable.

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d) If 0 is an eigenvalue, then $\operatorname{rank}(A) < n$.

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Question 12
Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

a) Find the eigenvalues and eigenspaces of A.

b) Show that A is diagonalizable and provide a formula for A^k for any $k \in \mathbb{N}$.

Question 13 Let $A, P \in M_{n\times n}$ with P invertible and $\lambda \in \mathbb{R}$.
Prove that λ is an eigenvalue of A if and only if λ is an eigenvalue of $P^{-1}AP$.

Hint: Simplify $P^{-1}(\lambda I_n)P$ and work with $\det(P^{-1}AP - \lambda I_n)$.

NB: A and $B = P^{-1}AP$ are called **similar matrices**.