Ens: Zsolt Patakfalvi - Analysis I - (n/a)

21 November 2018, 8:15 - duration: 60 mins

n/a

n/a

SCIPER: **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 4 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give

  +3 points if your answer is correct,

  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.
- For the **true/false** questions, we give

  +1 points if your answer is correct,

  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $(a_k)_{k\geq 0}$ be a sequence of real numbers and $s_n = \sum_{k=0}^{n} a_k$, $n \geq 0$ be the sequence of its partial sums. If $\lim_{n\to+\infty} s_n = 1$, then

- ☐ $\lim_{n\to+\infty} a_n = 1$
- ☐ $\lim_{n\to+\infty} s_{2n} < 1$
- ☐ $\lim_{n\to+\infty} (s_{2n} - 2s_n) = 0$
- ☐ $\lim_{n\to+\infty} (s_{2n} - s_n) = 0$

Question 2 : Let E be the subset of $\mathbb{R}$ defined by

$$E = \left\{ \sin\left(\frac{\pi n}{4}\right) - \sin\left(\frac{\pi}{4n}\right) : n \in \mathbb{N}\backslash\{0\} \right\} .$$

Then

- ☐ $\inf E = 0$
- ☐ $\inf E = -1$
- ☐ $\inf E = -1 - \sin\left(\frac{\pi}{24}\right)$
- ☐ $\inf E = -\frac{\sqrt{2}}{2}$

Question 3 : Let $(x_n)_{n\geq 0}$ be the sequence defined by $x_n = \dfrac{\sin(n\frac{\pi}{2})}{3 + \sin(n\frac{\pi}{2})}$. Then

- ☐ $\lim_{n\to\infty} x_n = \frac{1}{3}$
- ☐ $\lim_{n\to\infty} x_n = 0$
- ☐ $\liminf_{n\to\infty} x_n = -\frac{1}{2}$
- ☐ $\limsup_{n\to\infty} x_n = 1$

Question 4 : Let z be the complex number defined by $z = \mathrm{e}^{\mathrm{i}} + \mathrm{e}^{\mathrm{i}/3}$. Then

- ☐ $|z| = \sqrt{2 + 2\left(\mathrm{e}^{\mathrm{i}/3} + \mathrm{e}^{-\mathrm{i}/3}\right)}$
- ☐ $|z| = \sqrt{2}$
- ☐ $|z| = \sqrt{2 + 2\cos(\frac{2}{3})}$
- ☐ $|z| = \sqrt{1 + \left(\mathrm{e}^{2\mathrm{i}/3} + \mathrm{e}^{-2\mathrm{i}/3}\right)}$

Question 5 :
Let S be the series with parameter $c \in \mathbb{R}$, defined by

$$S = \sum_{n=1}^{\infty} \frac{n!}{n^{cn}} .$$

Then

- ☐ S converges if and only if $c > 3$
- ☐ S converges if and only if $c \geq 1$
- ☐ S converges if and only if $2 > c > 0$
- ☐ S converges if and only if $c \geq 0$

Question 6 : Consider a function $g \colon \mathbb{R} \to \mathbb{R}$ and the sequence of real numbers $(a_n)_{n\geq 0}$ defined recursively by $a_0 = 1$ and, for $n \geq 1$, $a_n = g(a_{n-1})$. Then the sequence $(a_n)_{n\geq 0}$ converges when g is defined by

- ☐ $g(x) = \frac{1}{4}x^2 + 1$
- ☐ $g(x) = 2x - 2$
- ☐ $g(x) = -x^2 + 2x - 2$
- ☐ $g(x) = x + 1$

Question 7 : Let $(a_n)_{n\geq 0}$ be an unbounded sequence of real numbers. Then

- ☐ for all $M > 0$, $|a_n| > M$ holds for all $n \geq 0$
- ☐ there exists n such that $a_n = +\infty$ or $a_n = -\infty$
- ☐ for all $M > 0$ there exists n such that $|a_n| \geq M$
- ☐ either $\lim_{n\to\infty} a_n = +\infty$, or $\lim_{n\to\infty} a_n = -\infty$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 8 : Let $(a_n)_{n\geq 0}$ be a sequence of real numbers such that the sequence $(b_n)_{n\geq 0}$ defined by $b_n = \cos(a_n)$ converges. Then the sequence $(a_n)_{n\geq 0}$ converges.

☐ TRUE ☐ FALSE

Question 9 : The series $\sum_{n=0}^{\infty} \left(\frac{x^2 - 1}{x^2 + 1}\right)^n$ converges for all $x \in \mathbb{R}$.

☐ TRUE ☐ FALSE

Question 10 : Let $(x_n)_{n\geq 1}$ be defined by $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$, and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x^2}{x^2+1}$. Then $\lim_{n\to\infty} f(x_n) = 1$.

☐ TRUE ☐ FALSE

Question 11 : Let A be a bounded subset of $\mathbb{R}$, and $c = \sup A$. Then for all $\epsilon > 0$ there exists $x \in A$ such that $x + \epsilon \geq c$.

☐ TRUE ☐ FALSE

Third part, open question

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes emtyp, they are used for the grading.

Question 12: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 *Do not write here.*

Consider the Fibonacci sequence, which is given by the recursive formula

$$x_n = x_{n-1} + x_{n-2}$$

and by the initial values

$$x_0 = x_1 = 1.$$

Recall that in class we showed that

$$\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = \frac{1+\sqrt{5}}{2}.$$

(a) Show by induction that $x_n \geq n$ for all $n \in \mathbb{N}$ [2 pts].

(b) Show that $\lim\limits_{n\to\infty} \sqrt[x_n]{n} = 1$ [3 pts].