Teach. Z. Patakfalvi - Analysis I - (n/a)

January 14, 2019 - duration: 3 hours

n/a

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SCIPER: **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 12 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give

  +3 points if your answer is correct,
  0 points if you give no answer or more than one,
  −1 points if your answer is incorrect.
- For the **true/false** questions, we give

  +1 points if your answer is correct,
  0 points if you give no answer or more than one,
  −1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : The improper integral $I = \int_{0^+}^{\frac{\pi}{6}} \frac{\cos(x)}{\sqrt{\sin(x)}}\,\mathrm{d}x$

- ☐ converges, and its value is $I = \sqrt{2}$
- ☐ diverges, because $\lim_{\varepsilon\to 0^+} \mathrm{Log}\big(\sqrt{\sin(\varepsilon)}\big) = -\infty$
- ☐ diverges, because $\frac{\cos(x)}{\sqrt{\sin(x)}}$ is not defined at $x = 0$
- ☐ converges, and its value is $I = \frac{1}{2}\,\mathrm{Log}(\frac{1}{2})$

Question 2 : Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x\,|\cos(x)|$. Then:

- ☐ f is not twice differentiable at $x = 0$
- ☐ f is infinitely many times differentiable on $\mathbb{R}$
- ☐ f is continuous on $\mathbb{R}$, but not differentiable at $x = 0$
- ☐ f is differentiable at $x = 0$, but not at $x = \frac{\pi}{2} + k\pi$, for every $k \in \mathbb{Z}$

Question 3 : Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \mathrm{e}^{\frac{x^4}{4}+\frac{x^2}{2}}$. Then:

- ☐ f has a single local maximum
- ☐ f has a single local minimum
- ☐ f is strictly increasing
- ☐ f is strictly decreasing

Question 4 : Set $\lambda := -\frac{1}{6}$. Find the convergent one out of the following series:

- ☐ $\sum_{n=1}^{\infty}\left(\frac{\lambda+1}{\lambda-1}\right)^n$
- ☐ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\lambda^n}$
- ☐ $\sum_{n=1}^{\infty}\left(\frac{1}{1-\lambda^2}\right)^n$
- ☐ $\sum_{n=1}^{\infty}\frac{1}{n^{1+\lambda}}$

Question 5 : Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = \frac{3}{2}$, and $a_{n+1} = \frac{1}{2} + \frac{1}{2}\sqrt{8a_n - 7}$, for every $n \geq 0$. Then:

- ☐ $\lim_{n\to\infty} a_n = 1$
- ☐ the sequence is divergent
- ☐ $\lim_{n\to\infty} a_n = +\infty$
- ☐ $\lim_{n\to\infty} a_n = 2$

Question 6 : Let $m \in \mathbb{R}$ be a real number, an let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{\sin^2(x)}{\mathrm{Log}\big(1+2x^2\big)} & \text{if } x < 0\,, \\ m & \text{if } x = 0\,, \\ \dfrac{x+1}{x^2+3x+1} & \text{if } x > 0\,. \end{cases}$$

- ☐ If $m = 1$, then f is continuous at $x = 0$.
- ☐ If $m = \frac{1}{3}$, then f is right continuous but not left continuous at $x = 0$.
- ☐ If $m = \frac{1}{2}$, then f is continuous at $x = 0$.
- ☐ If $m = \frac{1}{2}$, then f is left continuous but not right continuous at $x = 0$.

Question 7 : Consider the integral $I = \displaystyle\int_2^3 \frac{x^2-2x+1}{x^2+2x+1}\,\mathrm{d}x$. Then:

- ☐ $I = \frac{5}{3} - 4\,\mathrm{Log}(\frac{3}{2})$
- ☐ $I = 2\,\mathrm{Log}(2) + 1$
- ☐ $I = \mathrm{Log}(2) + \frac{1}{2}$
- ☐ $I = \frac{4}{3} - 4\,\mathrm{Log}(\frac{4}{3})$

Question 8 : Let $f :]-3,2[\ \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 4x - 1$. Then for every $x \in\ ]-3,2[$ and every $y \in\ ]-3,2[$ such that $x \neq y$ we have:

- ☐ $-3 \le \dfrac{f(x)-f(y)}{x-y} \le 7$
- ☐ $-2 \le \dfrac{f(x)-f(y)}{x-y} \le 8$
- ☐ $-4 \le \dfrac{f(x)-f(y)}{x-y} \le 6$
- ☐ $-1 \le \dfrac{f(x)-f(y)}{x-y} \le 9$

Question 9 : Let $p \in \mathbb{R}$ be a real number, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} |x|^p\,\mathrm{Log}\,|x|, & x \neq 0\,, \\ 0 & x = 0\,. \end{cases}$$

- ☐ If $p = \frac{6}{5}$, then f is differentiable at $x = 0$.
- ☐ If $p = \frac{3}{2}$, then f is not differentiable at $x = 0$.
- ☐ If $p = \frac{1}{2}$, then f is not continuous at $x = 0$.
- ☐ If $p = \frac{2}{3}$, then f is right continuous, but not left continuous at $x = 0$.

Question 10 : Let $S \subseteq \mathbb{C}$ be the set of solutions of the equation $\overline{z}^2 = z^2$ among the complex numbers. Then:

- ☐ $S = \mathbb{R}$
- ☐ $S = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0 \text{ or } \mathrm{Im}(z) = 0\}$
- ☐ $S = \{-1, +1, -\mathrm{i}, +\mathrm{i}\}$
- ☐ $S = \emptyset$

Question 11 : Let s be a real number, and let $(b_n)_{n\ge1}$ be the sequence defined by $b_n = \dfrac{1}{n^s}$ if n is even, and $b_n = \dfrac{1}{n^{2s}}$ if n is odd. Then the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converges if and only if

- ☐ $s > 2$
- ☐ $s > \frac{1}{2}$
- ☐ $s > 1$
- ☐ $s > 0$

Question 12 : The Taylor expansion of order 4 around 0 of the function $f(x) = \dfrac{1}{1-\sin(x)}$ is

- ☐ $1 + x + x^2 + \frac{5}{6}x^3 + \frac{2}{3}x^4$
- ☐ $1 + x + x^2 + x^3 + x^4$
- ☐ $1 + x + 2x^2 + 3x^3 + 4x^4$
- ☐ $1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4$

Question 13 : Let $(x_n)_{n\ge1}$ be the sequence defined by $x_n = \sqrt[n]{7}$ if n is even and by $x_n = \dfrac{1}{n^7}$ if n is odd. Then:

- ☐ $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n = 1$
- ☐ $\limsup\limits_{n\to\infty} x_n = 1$, and $\liminf\limits_{n\to\infty} x_n = 0$
- ☐ $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n = 0$
- ☐ $\limsup\limits_{n\to\infty} x_n = 0$, and $\liminf\limits_{n\to\infty} x_n = 1$

Question 14 : Let $f : \mathbb{R} \setminus \{-\frac{2}{3}\} \to \mathbb{R}$ be the function defined by $f(x) = \dfrac{4}{2+3x}$. The Taylor series of f around $x = 2$ is:

- ☐ $f(x) = \displaystyle\sum_{k=0}^{\infty} \left(-\tfrac{3}{8}\right)^k (x-2)^k$ for $x \in \left]1, 3\right[$
- ☐ $f(x) = \displaystyle\sum_{k=0}^{\infty} \left(\tfrac{3}{8}\right)^k (x-2)^k$ for $x \in \left]-\tfrac{2}{3}, \tfrac{14}{3}\right[$
- ☐ $f(x) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty} \left(-\tfrac{3}{8}\right)^k (x-2)^k$ for $x \in \left]-\tfrac{2}{3}, \tfrac{14}{3}\right[$
- ☐ $f(x) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty} \left(\tfrac{3}{8}\right)^k (x+2)^k$ for $x \in \left]-\tfrac{14}{3}, \tfrac{2}{3}\right[$

Question 15 : Let $(u_n)_{n\geq 0}$ be the sequence defined by $u_0 = 0$, and $u_{n+1} = \dfrac{1+2u_n}{2+u_n}$, for every $n \geq 0$. Then:

- ☐ $(u_n)_{n\geq 0}$ is decreasing
- ☐ $0 < u_n \leq 1$ for every $n \in \mathbb{N}^\star$
- ☐ $\lim\limits_{n\to\infty} u_n = \frac{1}{2}$
- ☐ $\frac{1}{2} < u_n \leq 1$ for every $n \in \mathbb{N}$

Question 16 : Consider the integral $I = \displaystyle\int_1^3 \frac{1}{\sqrt{x}\,(1+x)}\,\mathrm{d}x$. Then:

- ☐ $I = \sqrt{3} - 1 + \mathrm{Log}(2)$
- ☐ $I = \frac{1}{2}\left(\mathrm{Arctg}(3) - \frac{\pi}{4}\right)$
- ☐ $I = 2\left(\sqrt{3} - 1\right) + \mathrm{Log}(2)$
- ☐ $I = 2\left(\mathrm{Arctg}\left(\sqrt{3}\right) - \frac{\pi}{4}\right)$

Question 17 : Let A be the subset of $\mathbb{R}$ defined by $A = \left\{x > 0 \,:\, \cos\left(\dfrac{1}{x}\right) > 0\right\}$. Then

- ☐ $\mathrm{Inf}\, A = 0$
- ☐ $\mathrm{Inf}\, A = \dfrac{2}{\pi}$
- ☐ $\mathrm{Sup}\, A = 0$
- ☐ $\mathrm{Sup}\, A = \dfrac{\pi}{2}$

Question 18 : Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_n = \dfrac{2^{2n}}{(7n)!}$. As $n \to \infty$, this sequence

- ☐ converges to 0
- ☐ diverges
- ☐ converges to $\dfrac{4}{7}$
- ☐ converges to $\dfrac{\mathrm{Log}\,(2)}{7}$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $f :]-1,1[\to \mathbb{R}$ be a C^5 function, such that its Taylor expansion of order 4 around $x = 0$ is

$$f(x) = 1 + x - x^2 + x^3 - x^4 + x^4\varepsilon(x),$$

where $\lim_{x\to 0} \varepsilon(x) = 0$. Then $f'(0) + 3f^{(2)}(0) + f^{(3)}(0) = 1$.

☐ TRUE ☐ FALSE

Question 20 : Let $(x_n)_{n\geq 0}$ be the sequence defined by $x_0 = 2$, and $x_n = x_{n-1} - \dfrac{1}{n}$, for every $n \geq 1$. Then $(x_n)_{n\geq 0}$ is convergent.

☐ TRUE ☐ FALSE

Question 21 : The power series $\displaystyle\sum_{k=100}^{\infty} \frac{x^k}{k!}$ converges for every $x \in \mathbb{R}$.

☐ TRUE ☐ FALSE

Question 22 : Let $f\colon [-2,20] \to [0,1]$ be a continuous function. Then there is an $x \in [0,1]$ such that $f(x) = x$.

☐ TRUE ☐ FALSE

Question 23 : The series $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges.

☐ TRUE ☐ FALSE

Question 24 : Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function, such that there is exactly one solution of the equation $f'(x) = 0$. Then the equation $f(x) = 1$ has at most two distinct real solutions.

☐ TRUE ☐ FALSE

Question 25 : Let $A \subset \mathbb{R}$ be a bounded set, and let $B := \{x \in \mathbb{R} : x \text{ is an upper bound of } A\}$. Then $\operatorname{Inf} B \in B$.

☐ TRUE ☐ FALSE

Question 26 : Let $z \neq 0$ be a complex number with argument $\frac{\pi}{4}$. Then, the argument of the complex number $\frac{1}{z^2}$ is $-\frac{\pi}{2}$.

☐ TRUE ☐ FALSE

Question 27 : The function $g : [-1, 1] \to \mathbb{R}$ defined by $g(x) = \int_0^{|x|} 1 \, dt$ is differentiable at $x = 0$.

☐ TRUE ☐ FALSE

Question 28 : Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective function. Then f is strictly monotone.

☐ TRUE ☐ FALSE

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 29: *This question is worth 10 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 *Do not write here.*

Consider the function

$$f(x) := \frac{\sin^3(x)}{6} + \frac{\sin^2(x)}{8} + \frac{1}{96} : \left[-\frac{\pi}{2}, \frac{\pi}{6}\right] \to \mathbb{R}$$

(a) Show that

$$\max_{x\in[-\frac{\pi}{2},\frac{\pi}{6}]} f(x) = \frac{1}{16} \qquad \text{and} \qquad \min_{x\in[-\frac{\pi}{2},\frac{\pi}{6}]} f(x) = -\frac{1}{32}.$$

(b) How many $x \in \left[-\frac{\pi}{2}, \frac{\pi}{6}\right]$ are there such that $f(x) = 0$ holds?

Question 30: *This question is worth 6 points.*

0 1 2 3 4 5 6 *Do not write here.*

Consider the following power series

$$\sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}+n}(x-3)^n.$$

(a) Find its radius of convergence.

(b) Determine if the power series is convergent at the end-points of the interval of convergence.