EPFL

Lecturer: Z. Patakfalvi
Analysis I - (n/a)
13th January 2020
3 hours

n/a

n/a

SCIPER: **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 16 pages, the last ones possibly blank. TOTAL: 31 questions. Do not unstaple.

- Place your student card on your table.
- **The only papers you are allowed to use are the booklet of the exam and the scratch paper provided by the proctors.**
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  - $+3$ points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - -1 points if your answer is incorrect.
- For the **true/false** questions, we give :
  - $+1$ points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - -1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Set $I = \int_0^2 e^{(x^2)}\,\mathrm{d}x$. Then,

- ☐ $2 \le I \le 200$.
- ☐ $I = e^{\frac{8}{3}} - 1$.
- ☐ $0 \le I < \frac{14}{3}$.
- ☐ $I \ge 200$.

Question 2 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x \sin\left(\mathrm{e}^{\frac{1}{x}} - 1\right) & \text{if} \quad x \neq 0\,, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Then,

- ☐ f is continuous over $\mathbb{R}$, and it is differentiable from the right, but not from the left at $x = 0$.
- ☐ f is C^1 over $\mathbb{R}$, that is, it is differentiable and its derivative is continuous over $\mathbb{R}$.
- ☐ f is differentiable over $\mathbb{R}$, but f' is not continuous over the entire $\mathbb{R}$.
- ☐ f is continuous over $\mathbb{R}$, and it is differentiable from the left but not from the right at $x = 0$.

Question 3 : Define $a_n = \left(\sqrt{n+2} - \sqrt{n+1}\right) \sin\left(\frac{1}{n}\right)$, for every $n \in \mathbb{N}^*$. Then,

- ☐ the series $\sum_{n=1}^{\infty} a_n$ is convergent, but it is not absolutely convergent.
- ☐ the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- ☐ both the series $\sum_{n=1}^{\infty} a_n$ and the series $\sum_{n=1}^{\infty} (-1)^n a_n$ are convergent.
- ☐ the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is divergent.

Question 4 : Let R be the radius of convergence of the power series $f(x) = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)^{(n^b)} x^n$.

- ☐ If $b = 2$, then $R = 1$.
- ☐ If $b = 1$, then $R = e^{-1}$.
- ☐ If $b = 3$, then $R = e$.
- ☐ If $b = 4$, then $R = e^2$.

Question 5 : Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = |x \cos(x)|$.

- ☐ There exists $u \in \left]0, \frac{\pi}{4}\right[$ such that $f'(u) = \frac{\sqrt{2}}{2}$.
- ☐ f is increasing on $\left]0, \frac{\pi}{2}\right[$.
- ☐ There is a single local minimum of f on the entire $\mathbb{R}$.
- ☐ There exists $u \in \left]-\frac{\pi}{8}, \frac{\pi}{8}\right[$ such that $f'(u) = 0$.

Question 6 : Among the functions $f, g, h : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x}\,\sin(\frac{1}{x}) & \text{if } x > 0 \\ -\sqrt{-x} & \text{if } x \le 0 \end{cases}, \qquad g(x) = \begin{cases} x\,\mathrm{sh}(\frac{1}{x}) & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases},$$

$$h(x) = \begin{cases} \sqrt{x}\,\mathrm{Arctg}(\frac{1}{x}) & \text{si } x > 0 \\ x\,\mathrm{Log}(|x|) & \text{si } x < 0 \\ 0 & \text{si } x = 0 \end{cases},$$

find those that are continuous at $x = 0$:

☐ g and h. ☐ f and g. ☐ all three. ☐ f and h.

Question 7 : Consider the integral $I = \displaystyle\int_1^2 x\,\mathrm{Log}(1+x)\,\mathrm{d}x$. Then,

☐ $I = 2\,\mathrm{Log}(3) - \frac{1}{2}\,\mathrm{Log}(2)$. ☐ $I = \frac{1}{2}\,\mathrm{Log}(2) + \frac{1}{4}$.

☐ $I = 2\,\mathrm{Log}(3) + \frac{1}{2}\,\mathrm{Log}(2)$. ☐ $I = \frac{3}{2}\,\mathrm{Log}(3) - \frac{1}{4}$.

Question 8 : Let $f\colon [1, +\infty[\to \mathbb{R}$ be the function defined by $f(x) = \sin\left(\mathrm{Arctg}(\sqrt{x}\,)\right)$. Then the range of f is

☐ $]0, 1]$. ☐ $[-1, 1]$. ☐ $\left[\frac{\sqrt{2}}{2}, 1\right[$. ☐ $\left[0, \frac{\sqrt{2}}{2}\right[$.

Question 9 : For which numbers $a, b \in \mathbb{R}$ is the function $f\colon \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} (ax+1)(bx-1) & \text{if } x \ge 0, \\ \sin(a^2 x) - b & \text{if } x < 0, \end{cases}$$

differentiable at $x = 0$?

☐ $a = \dfrac{1 \pm \sqrt{5}}{2}$ and $b = -1$ ☐ $a = \dfrac{-1 \pm \sqrt{5}}{2}$ and $b = 1$

☐ $a = \pm 1$ and $b = -1$ ☐ $a = \pm 1$ and $b = 1$

Question 10 : The imaginary part of $\left(-1 + i\sqrt{3}\,\right)^5$ is

☐ $16\sqrt{3}$. ☐ $-16\sqrt{3}$. ☐ $32\sqrt{3}$. ☐ $32\sqrt{3}\,i$.

Question 11 : The limit $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{5n + \sqrt{3n - \sqrt{2n}}}}$

☐ exists, and it is $\frac{1}{\sqrt{5+\sqrt{3-\sqrt{2}}}}$. ☐ exists, and it is $\frac{1}{\sqrt{5}}$.

☐ exists, and it is $\frac{1}{\sqrt{6}}$. ☐ does not exist.

Question 12 : Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{1}{1+\mathrm{e}^{-x}}$. The Taylor expansion of order 3 of f around $x_0 = 0$ is

- ☐ $f(x) = \dfrac{1}{2} - \dfrac{x}{4} - \dfrac{x^3}{48} + x^3\varepsilon(x)$.
- ☐ $f(x) = \dfrac{1}{2} + \dfrac{x}{4} + \dfrac{x^3}{48} + x^3\varepsilon(x)$.
- ☐ $f(x) = \dfrac{1}{2} + \dfrac{x}{4} - \dfrac{x^3}{48} + x^3\varepsilon(x)$.
- ☐ $f(x) = \dfrac{1}{2} + \dfrac{x}{4} - \dfrac{x^3}{24} + x^3\varepsilon(x)$.

Question 13 : Let $(a_n)_{n\geq 1}$ be the sequence defined as follows: for all $n \geq 1$,

$$a_n = \sin\left(\frac{\pi}{4} + n\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4} + n\frac{\pi}{2}\right).$$

Then,

- ☐ $\limsup_{n\to\infty} a_n = 2$ and $\liminf_{n\to\infty} a_n = -2$.
- ☐ $\limsup_{n\to\infty} a_n = \sqrt{2}$ and $\liminf_{n\to\infty} a_n = -\sqrt{2}$.
- ☐ $\limsup_{n\to\infty} a_n = 0$ and $\liminf_{n\to\infty} a_n = -\sqrt{2}$.
- ☐ $\limsup_{n\to\infty} a_n = \sqrt{2}$ and $\liminf_{n\to\infty} a_n = 0$.

Question 14 : The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^{\frac{2}{\alpha}}\left(n^{2\alpha}+1\right)}}$ converges if

- ☐ $\frac{1}{2} < \alpha < 1$.
- ☐ $\alpha = \frac{1}{2}$.
- ☐ $0 < \alpha < \frac{1}{2}$.
- ☐ $1 < \alpha < 2$.

Question 15 : Set $x_0 \in \mathbb{R}$, and let $x_{n+1} = x_n - \dfrac{1}{3^n}$ for every $n \in \mathbb{N}$. Then

- ☐ for all $x_0 \in \mathbb{R}$, the sequence $(x_n)_{n\geq 0}$ converges to x_0.
- ☐ for all $x_0 \in \mathbb{R}$, the sequence $(x_n)_{n\geq 0}$ converges to 0.
- ☐ for all $x_0 \in \mathbb{R}$, the sequence $(x_n)_{n\geq 0}$ converges to $x_0 - \frac{3}{2}$.
- ☐ for all $x_0 \in \mathbb{R}$, the sequence $(x_n)_{n\geq 0}$ is divergent.

Question 16 : The improper integral $\displaystyle\int_1^{\infty} \frac{x^{3/2}+3}{x^3}\,dx$

- ☐ diverges.
- ☐ converges, and its value is $-\frac{7}{2}$.
- ☐ converges, and its value is $\frac{8}{3}$.
- ☐ converges, and its value is $\frac{7}{2}$.

Question 17 : Set $A = \left\{x \in \mathbb{R}_+^* \setminus \{1\} \,:\, \dfrac{1}{\mathrm{Log}(x)} < 1\right\}$. Then,

- ☐ $\mathrm{Inf}\, A = 0$.
- ☐ A is not bounded from below.
- ☐ $\mathrm{Sup}\, A = \mathrm{e}$.
- ☐ $\mathrm{Inf}\, A = \mathrm{e}$.

Question 18 : Define $f\colon \mathbb{R}^* \to \mathbb{R}$ by $f(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$, and define the sequence $(x_n)_{n\geq 1}$ by setting $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$, and for some fixed $x_0 \in \mathbb{R}^*$.

- ☐ If $x_0 = -2$, then the sequence $(x_n)_{n\geq 1}$ converges to $-\sqrt{2}$.
- ☐ If $x_0 = \frac{1}{\sqrt{2}}$, then the sequence $(x_n)_{n\geq 1}$ converges to $-\sqrt{2}$.
- ☐ If $x_0 = 1$, then the sequence $(x_n)_{n\geq 1}$ converges to $-\sqrt{2}$.
- ☐ There does not exist any $x_0 \in \mathbb{R}^*$ for which the sequence $(x_n)_{n\geq 1}$ converges to $-\sqrt{2}$.

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Then f is continuous at exactly two points.

☐ TRUE ☐ FALSE

Question 20 : A strictly increasing function $f : [0,1] \to [0,1]$ is always bijective.

☐ TRUE ☐ FALSE

Question 21 : The radius of convergence of the power series $f(x) = \sum_{k=0}^{+\infty} (3x)^k$ is 3.

☐ TRUE ☐ FALSE

Question 22 : Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function, and let $x_0 \in \mathbb{R}$ be such that

$$\lim_{x \to x_0-} f(x) = f(x_0).$$

Then f is differentiable from the left at x_0.

☐ TRUE ☐ FALSE

Question 23 : Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers. If $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

☐ TRUE ☐ FALSE

Question 24 : Consider a function $f \colon \mathbb{R} \to \mathbb{R}$. If f is differentiable at x_0, then the function $g \colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sin(f(x))$ is also differentiable at x_0.

☐ TRUE ☐ FALSE

Question 25 : Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and increasing function, and for all $n \in \mathbb{N}$, let a_n be the real number defined by $a_n = f(n)$. Then $(a_n)_{n\geq 0}$ is a Cauchy sequence.

☐ TRUE ☐ FALSE

Question 26 : For all $\omega \in \mathbb{C}$, $\omega \neq 0$, there exist infinitely many complex numbers $z \in \mathbb{C}$ such that $\operatorname{Im}(\omega z) = 0$.

☐ TRUE ☐ FALSE

Question 27 : Let $f\colon [-1,1] \to [-1,1]$ be a continuous and bijective function, such that $f(0) = 0$. Then $\displaystyle\int_{-1}^{1} f(x)\,dx = 0$.

☐ TRUE ☐ FALSE

Question 28 : Let $f\colon \,]-1,1[\, \to \mathbb{R}$ be a C^3 function, that is, the third derivative of f exists and it is continuous. Assume that the Taylor expansion of order 2 of f around $x_0 = 0$ is given by $f(x) = 1 + 2x + x^2 + x^2\varepsilon_1(x)$. Then the Taylor expansion of the function $(f(x))^2$ around $x_0 = 0$ is $(f(x))^2 = 1 + 4x^2 + x^2\varepsilon_2(x)$.

☐ TRUE ☐ FALSE