Teach. Z. Patakfalvi - Analysis I - (n/a)

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

January 14, 2019 - duration: 3 hours

n/a

n/a

SCIPER: **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 12 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give

  +3 points if your answer is correct,

  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.
- For the **true/false** questions, we give

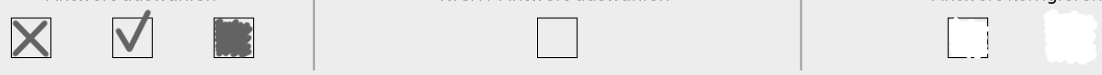
  +1 points if your answer is correct,

  0 points if you give no answer or more than one,

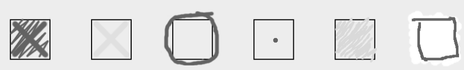
  −1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen — ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen — Corriger une réponse | Correct an answer Antwort korrigieren



ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte



First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question [`QCM-complexes-B`] **:** Let $S \subseteq \mathbb{C}$ be the set of solutions of the equation $\overline{z}^2 = z^2$ among the complex numbers. Then:

- [ ] $S = \{-1, +1, -\mathrm{i}, +\mathrm{i}\}$
- [ ] $S = \emptyset$
- [x] $S = \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) = 0 \text{ or } \mathrm{Im}(z) = 0\}$
- [ ] $S = \mathbb{R}$

Question [`QCM-contin-deriv-C1-B`] **:** Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x\,|\cos(x)|$. Then:

- [ ] f is continuous on $\mathbb{R}$, but not differentiable at $x = 0$
- [x] f is differentiable at $x = 0$, but not at $x = \frac{\pi}{2} + k\pi$, for every $k \in \mathbb{Z}$
- [ ] f is not twice differentiable at $x = 0$
- [ ] f is infinitely many times differentiable on $\mathbb{R}$

Question [`QCM-contin-vs-derivab-B`] **:** Let $p \in \mathbb{R}$ be a real number, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} |x|^p \operatorname{Log}|x|, & x \neq 0\,, \\ 0 & x = 0\,. \end{cases}$$

- [x] If $p = \frac{6}{5}$, then f is differentiable at $x = 0$.
- [ ] If $p = \frac{1}{2}$, then f is not continuous at $x = 0$.
- [ ] If $p = \frac{3}{2}$, then f is not differentiable at $x = 0$.
- [ ] If $p = \frac{2}{3}$, then f is right continuous, but not left continuous at $x = 0$.

Question [`QCM-dev-limite-A`] **:** The Taylor expansion of order 4 around 0 of the function $f(x) = \dfrac{1}{1 - \sin(x)}$ is

- [ ] $1 + x + 2x^2 + 3x^3 + 4x^4$
- [x] $1 + x + x^2 + \frac{5}{6}x^3 + \frac{2}{3}x^4$
- [ ] $1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4$
- [ ] $1 + x + x^2 + x^3 + x^4$

Question [`QCM-induction-A-2`] **:** Let $(u_n)_{n\geq 0}$ be the sequence defined by $u_0 = 0$, and $u_{n+1} = \dfrac{1 + 2u_n}{2 + u_n}$, for every $n \geq 0$. Then:

- [x] $0 < u_n \leq 1$ for every $n \in \mathbb{N}^\star$
- [ ] $(u_n)_{n\geq 0}$ is decreasing
- [ ] $\lim\limits_{n\to\infty} u_n = \frac{1}{2}$
- [ ] $\frac{1}{2} < u_n \leq 1$ for every $n \in \mathbb{N}$

Question [`QCM-inf-sup-E`] **:** Let A be the subset of $\mathbb{R}$ defined by $A = \left\{x > 0 \,:\, \cos\left(\dfrac{1}{x}\right) > 0\right\}$. Then

- [x] $\operatorname{Inf} A = 0$
- [ ] $\operatorname{Sup} A = \dfrac{\pi}{2}$
- [ ] $\operatorname{Sup} A = 0$
- [ ] $\operatorname{Inf} A = \dfrac{2}{\pi}$

CATALOG

Question [`QCM-int-generalisee-A`] : The improper integral $I = \displaystyle\int_{0^+}^{\frac{\pi}{6}} \frac{\cos(x)}{\sqrt{\sin(x)}}\,\mathrm{d}x$

■ converges, and its value is $I = \sqrt{2}$

□ converges, and its value is $I = \frac{1}{2}\,\mathrm{Log}(\frac{1}{2})$

□ diverges, because $\displaystyle\lim_{\varepsilon\to 0^+} \mathrm{Log}\big(\sqrt{\sin(\varepsilon)}\,\big) = -\infty$

□ diverges, because $\dfrac{\cos(x)}{\sqrt{\sin(x)}}$ is not defined at $x = 0$

Question [`QCM-integrale-first-A`] : Consider the integral $I = \displaystyle\int_2^3 \frac{x^2 - 2x + 1}{x^2 + 2x + 1}\,\mathrm{d}x$. Then:

■ $I = \frac{4}{3} - 4\,\mathrm{Log}(\frac{4}{3})$

□ $I = \mathrm{Log}(2) + \frac{1}{2}$

□ $I = \frac{5}{3} - 4\,\mathrm{Log}(\frac{3}{2})$

□ $I = 2\,\mathrm{Log}(2) + 1$

Question [`QCM-integrale-second-A`] : Consider the integral $I = \displaystyle\int_1^3 \frac{1}{\sqrt{x}\,(1+x)}\,\mathrm{d}x$. Then:

■ $I = 2\big(\mathrm{Arctg}(\sqrt{3}\,) - \frac{\pi}{4}\big)$

□ $I = 2\big(\sqrt{3} - 1\big) + \mathrm{Log}(2)$

□ $I = \frac{1}{2}\big(\mathrm{Arctg}(3) - \frac{\pi}{4}\big)$

□ $I = \sqrt{3} - 1 + \mathrm{Log}(2)$

Question [`QCM-limite-prolongmt-B`] : Let $m \in \mathbb{R}$ be a real number, an let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{\sin^2(x)}{\mathrm{Log}\big(1 + 2x^2\big)} & \text{if } x < 0\,, \\ m & \text{if } x = 0\,, \\ \dfrac{x+1}{x^2 + 3x + 1} & \text{if } x > 0\,. \end{cases}$$

■ If $m = \frac{1}{2}$, then f is left continuous but not right continuous at $x = 0$.

□ If $m = \frac{1}{3}$, then f is right continuous but not left continuous at $x = 0$.

□ If $m = 1$, then f is continuous at $x = 0$.

□ If $m = \frac{1}{2}$, then f is continuous at $x = 0$.

Question [`QCM-limsup-liminf-B`] : Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_n = \sqrt[n]{7}$ if n is even and by $x_n = \dfrac{1}{n^7}$ if n is odd. Then:

■ $\displaystyle\limsup_{n\to\infty} x_n = 1$, and $\displaystyle\liminf_{n\to\infty} x_n = 0$

□ $\displaystyle\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 0$

□ $\displaystyle\limsup_{n\to\infty} x_n = 0$, and $\displaystyle\liminf_{n\to\infty} x_n = 1$

□ $\displaystyle\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 1$

Question [`QCM-propriete-fonction-A`] : Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \mathrm{e}^{\frac{x^4}{4} + \frac{x^2}{2}}$. Then:

■ f has a single local minimum

□ f is strictly increasing

□ f has a single local maximum

□ f is strictly decreasing

Question [`QCM-serie-B`] : Set $\lambda := -\frac{1}{6}$. Find the convergent one out of the following series:

■ $\displaystyle\sum_{n=1}^{\infty} \Big(\frac{\lambda+1}{\lambda-1}\Big)^n$

□ $\displaystyle\sum_{n=1}^{\infty} \Big(\frac{1}{1-\lambda^2}\Big)^n$

□ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\lambda^n}$

□ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\lambda}}$

Question [QCM-serie-entiere-A] **:** Let $f : \mathbb{R}\backslash\{-\frac{2}{3}\} \to \mathbb{R}$ be the function defined by $f(x) = \dfrac{4}{2+3x}$. The Taylor series of f around $x = 2$ is:

- ■ $f(x) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty}\left(-\tfrac{3}{8}\right)^k (x-2)^k$ for $x \in \left]-\tfrac{2}{3}, \tfrac{14}{3}\right[$
- ☐ $f(x) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty}\left(\tfrac{3}{8}\right)^k (x+2)^k$ for $x \in \left]-\tfrac{14}{3}, \tfrac{2}{3}\right[$
- ☐ $f(x) = \displaystyle\sum_{k=0}^{\infty}\left(\tfrac{3}{8}\right)^k (x-2)^k$ for $x \in \left]-\tfrac{2}{3}, \tfrac{14}{3}\right[$
- ☐ $f(x) = \displaystyle\sum_{k=0}^{\infty}\left(-\tfrac{3}{8}\right)^k (x-2)^k$ for $x \in \left]1, 3\right[$

Question [QCM-serie-parametre-B] **:** Let s be a real number, and let $(b_n)_{n\geq 1}$ be the sequence defined by $b_n = \dfrac{1}{n^s}$ if n is even, and $b_n = \dfrac{1}{n^{2s}}$ if n is odd. Then the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converges if and only if

- ■ $s > 1$
- ☐ $s > \frac{1}{2}$
- ☐ $s > 0$
- ☐ $s > 2$

Question [QCM-suites-convergence-A] **:** Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_n = \dfrac{2^{2n}}{(7n)!}$. As $n \to \infty$, this sequence

- ■ converges to 0
- ☐ converges to $\dfrac{4}{7}$
- ☐ diverges
- ☐ converges to $\dfrac{\operatorname{Log}(2)}{7}$

Question [QCM-suites-recurrence-B] **:** Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = \frac{3}{2}$, and $a_{n+1} = \frac{1}{2} + \frac{1}{2}\sqrt{8a_n - 7}$, for every $n \geq 0$. Then:

- ☐ $\lim_{n\to\infty} a_n = 1$
- ☐ $\lim_{n\to\infty} a_n = +\infty$
- ■ $\lim_{n\to\infty} a_n = 2$
- ☐ the sequence is divergent

Question [QCM-theo-accr-finis-B] **:** Let $f : \left]-3, 2\right[\to \mathbb{R}$ be the function defined by $f(x) = x^2 + 4x - 1$. Then for every $x \in \left]-3, 2\right[$ and every $y \in \left]-3, 2\right[$ such that $x \neq y$ we have:

- ☐ $-1 \leq \dfrac{f(x) - f(y)}{x - y} \leq 9$
- ■ $-2 \leq \dfrac{f(x) - f(y)}{x - y} \leq 8$
- ☐ $-3 \leq \dfrac{f(x) - f(y)}{x - y} \leq 7$
- ☐ $-4 \leq \dfrac{f(x) - f(y)}{x - y} \leq 6$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question [`TF-complexes-C`] : Let $z \neq 0$ be a complex number with argument $\frac{\pi}{4}$. Then, the argument of the complex number $\frac{1}{z^2}$ is $-\frac{\pi}{2}$.

■ TRUE ☐ FALSE

Question [`TF-derivabilite-discussion-B`] : Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function, such that there is exactly one solution of the equation $f'(x) = 0$. Then the equation $f(x) = 1$ has at most two distinct real solutions.

■ TRUE ☐ FALSE

Question [`TF-dev-limite-B`] : Let $f :]-1, 1[\to \mathbb{R}$ be a C^5 function, such that its Taylor expansion of order 4 around $x = 0$ is

$$f(x) = 1 + x - x^2 + x^3 - x^4 + x^4\varepsilon(x),$$

where $\lim_{x\to 0} \varepsilon(x) = 0$. Then $f'(0) + 3f^{(2)}(0) + f^{(3)}(0) = 1$.

■ TRUE ☐ FALSE

Question [`TF-fonction-etc-B`] : Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective function. Then f is strictly monotone.

☐ TRUE ■ FALSE

Question [`TF-induction-suites-limites-A`] : Let $(x_n)_{n\geq 0}$ be the sequence defined by $x_0 = 2$, and $x_n = x_{n-1} - \frac{1}{n}$, for every $n \geq 1$. Then $(x_n)_{n\geq 0}$ is convergent.

☐ TRUE ■ FALSE

Question [`TF-inf-sup-B`] : Let $A \subset \mathbb{R}$ be a bounded set, and let $B := \{x \in \mathbb{R} : x \text{ is an upper bound of } A\}$. Then $\operatorname{Inf} B \in B$.

■ TRUE ☐ FALSE

Question [`TF-integrale-B`] : The function $g : [-1, 1] \to \mathbb{R}$ defined by $g(x) = \int_0^{|x|} 1\,dt$ is differentiable at $x = 0$.

☐ TRUE ■ FALSE

Question [`TF-limites-continuite-A`] **:** Let $f\colon [-2,20] \to [0,1]$ be a continuous function. Then there is an $x \in [0,1]$ such that $f(x) = x$.

■ TRUE ☐ FALSE

Question [`TF-serie-AA`] **:** The series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges.

■ TRUE ☐ FALSE

Question [`TF-serie-entiere-B`] **:** The power series $\sum_{k=100}^{\infty} \frac{x^k}{k!}$ converges for every $x \in \mathbb{R}$.

■ TRUE ☐ FALSE