Ens: Zsolt Patakfalvi - Analysis I - (n/a)

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

21 November 2018, 8:15 - duration: 60 mins

n/a

n/a

SCIPER: **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 4 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give

  +3 points if your answer is correct,

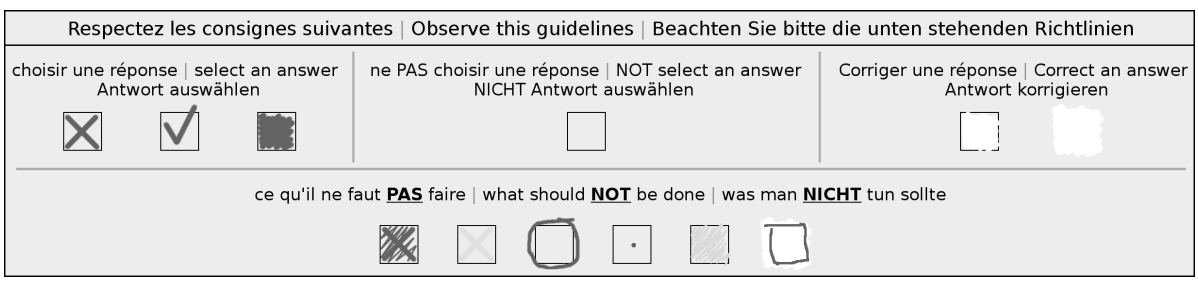
  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.
- For the **true**/**false** questions, we give

  +1 points if your answer is correct,

  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.



First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $(a_k)_{k\geq 0}$ be a sequence of real numbers and $s_n = \sum_{k=0}^{n} a_k$, $n \geq 0$ be the sequence of its partial sums. If $\lim_{n\to+\infty} s_n = 1$, then

☐ $\lim_{n\to+\infty} a_n = 1$
☐ $\lim_{n\to+\infty} s_{2n} < 1$
☐ $\lim_{n\to+\infty} (s_{2n} - 2s_n) = 0$
☐ $\lim_{n\to+\infty} (s_{2n} - s_n) = 0$

Question 2 : Let E be the subset of $\mathbb{R}$ defined by

$$E = \left\{ \sin\left(\frac{\pi n}{4}\right) - \sin\left(\frac{\pi}{4n}\right) : n \in \mathbb{N}\setminus\{0\} \right\} .$$

Then

☐ $\inf E = 0$
☐ $\inf E = -1$
☐ $\inf E = -1 - \sin\left(\frac{\pi}{24}\right)$
☐ $\inf E = -\frac{\sqrt{2}}{2}$

Question 3 : Let $(x_n)_{n\geq 0}$ be the sequence defined by $x_n = \frac{\sin(n\frac{\pi}{2})}{3+\sin(n\frac{\pi}{2})}$. Then

☐ $\lim_{n\to\infty} x_n = \frac{1}{3}$
☐ $\lim_{n\to\infty} x_n = 0$
☐ $\liminf_{n\to\infty} x_n = -\frac{1}{2}$
☐ $\limsup_{n\to\infty} x_n = 1$

Question 4 : Let z be the complex number defined by $z = \mathrm{e}^{\mathrm{i}} + \mathrm{e}^{\mathrm{i}/3}$. Then

☐ $|z| = \sqrt{2 + 2\left(\mathrm{e}^{\mathrm{i}/3} + \mathrm{e}^{-\mathrm{i}/3}\right)}$
☐ $|z| = \sqrt{2}$
☐ $|z| = \sqrt{2 + 2\cos(\frac{2}{3})}$
☐ $|z| = \sqrt{1 + \left(\mathrm{e}^{2\mathrm{i}/3} + \mathrm{e}^{-2\mathrm{i}/3}\right)}$

Question 5 :
Let S be the series with parameter $c \in \mathbb{R}$, defined by

$$S = \sum_{n=1}^{\infty} \frac{n!}{n^{cn}} .$$

Then

☐ S converges if and only if $c > 3$
☐ S converges if and only if $c \geq 1$
☐ S converges if and only if $2 > c > 0$
☐ S converges if and only if $c \geq 0$

Question 6 : Consider a function $g\colon \mathbb{R} \to \mathbb{R}$ and the sequence of real numbers $(a_n)_{n\geq 0}$ defined recursively by $a_0 = 1$ and, for $n \geq 1$, $a_n = g(a_{n-1})$. Then the sequence $(a_n)_{n\geq 0}$ converges when g is defined by

☐ $g(x) = \frac{1}{4}x^2 + 1$
☐ $g(x) = 2x - 2$
☐ $g(x) = -x^2 + 2x - 2$
☐ $g(x) = x + 1$

Question 7 : Let $(a_n)_{n\geq 0}$ be an unbounded sequence of real numbers. Then

- ☐ for all $M > 0$, $|a_n| > M$ holds for all $n \geq 0$
- ☐ there exists n such that $a_n = +\infty$ or $a_n = -\infty$
- ☐ for all $M > 0$ there exists n such that $|a_n| \geq M$
- ☐ either $\lim_{n\to\infty} a_n = +\infty$, or $\lim_{n\to\infty} a_n = -\infty$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 8 : Let $(a_n)_{n\geq 0}$ be a sequence of real numbers such that the sequence $(b_n)_{n\geq 0}$ defined by $b_n = \cos(a_n)$ converges. Then the sequence $(a_n)_{n\geq 0}$ converges.

☐ TRUE ☐ FALSE

Question 9 : The series $\displaystyle\sum_{n=0}^{\infty} \Big(\frac{x^2-1}{x^2+1}\Big)^n$ converges for all $x \in \mathbb{R}$.

☐ TRUE ☐ FALSE

Question 10 : Let $(x_n)_{n\geq 1}$ be defined by $x_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}$, and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x^2}{x^2+1}$. Then $\lim_{n\to\infty} f(x_n) = 1$.

☐ TRUE ☐ FALSE

Question 11 : Let A be a bounded subset of $\mathbb{R}$, and $c = \sup A$. Then for all $\epsilon > 0$ there exists $x \in A$ such that $x + \epsilon \geq c$.

☐ TRUE ☐ FALSE