Duration : 67 minutes

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Analysis I

Midterm test
Fall 2016

SCIPER: **12345678**

Do not turn the page before the start of the exam. This document is double-sided, has 4 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give

  +3 points if your answer is correct,

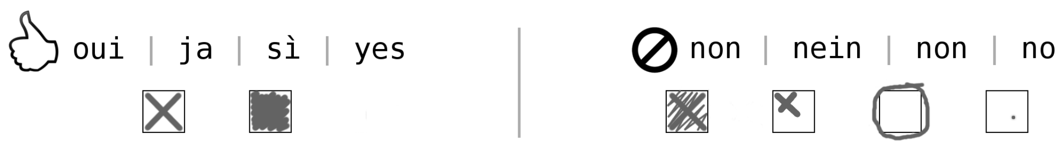
  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.
- For the **true/false** questions, we give

  +1 points if your answer is correct,

  0 points if you give no answer or more than one,

  −1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.
- Observe these guidelines when **recording your answers**:



First part: multiple choice questions

For each question, cross the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $\{x_n\}$ be the bounded sequence defined by $x_n = \dfrac{5 + 3^{n+1}}{1 + (-3)^n}$.

Then

- ☐ $\liminf_{n\to+\infty} x_n = -7$
- ☐ $\limsup_{n\to+\infty} x_n = 4$
- ☐ the sequence (x_n) converges
- ■ $\liminf_{n\to+\infty} x_n = -3$

Question 2 : The value of the limit $\lim_{n\to+\infty} \left(1 - \dfrac{2}{n} + \dfrac{1}{n^2}\right)^n$ is

- ☐ e
- ☐ 1
- ■ e^{-2}
- ☐ 0

Question 3 : Let $E \subset \mathbb{R}$ be the subset defined by $E = \left\{ \left(1 + \dfrac{1}{n}\right)^{-1} : n \in \mathbb{N} \setminus \{0\} \right\}$.

Then

- ■ $\operatorname{Sup} E = 1$ and $\operatorname{Inf} E = \dfrac{1}{2}$
- ☐ $\operatorname{Sup} E \notin E$ and $\operatorname{Inf} E \notin E$
- ☐ $\operatorname{Sup} E = 1$ and $\operatorname{Inf} E = 0$
- ☐ $\operatorname{Sup} E = 2$ and $\operatorname{Inf} E = \dfrac{1}{2}$

Question 4 : Let $n \in \mathbb{N} \setminus \{0\}$. Define the sum $S_n = \sum_{k=1}^{2n+1} (-1)^k\, k$.

Then we have

- ☐ $S_n = n$
- ■ $S_n = -n - 1$
- ☐ $S_n = -1$
- ☐ $S_n = -n$

Question 5 : Let $d \in \mathbb{N} \setminus \{0\}$ and consider the series $\sum_{n=0}^{+\infty} \dfrac{(n!)^d}{(d\cdot n)!}$.

- ☐ The series converges only for $d = 2$
- ☐ The series converges for all d
- ☐ The series diverges for all $d \leq 5$
- ■ The series converges for all $d \geq 2$

Question 6 : Let the function $f\colon \left[-\frac{1}{6}, \frac{1}{6}\right] \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = \dfrac{\sqrt{1+6x}-1}{\sin(2x)}$.
If it exists, let $g\colon \left[-\frac{1}{6}, \frac{1}{6}\right] \to \mathbb{R}$ be the extension of f by continuity at 0.
Then

- ☐ f does not admit an extension by continuity at 0
- ☐ g exists and $g(0) = 1$
- ■ g exists and $g(0) = \dfrac{3}{2}$
- ☐ g exists and $g(0) = \dfrac{1}{2}$

Question 7 : For all $x \in \mathbb{R}$ and all $y \in \mathbb{R}$ such that $x + \mathrm{i}y \neq \mathrm{i}$, the complex number $z = x + \mathrm{i}y$ satisfies

- ☐ $\mathrm{Re}\left(\dfrac{z^2}{\mathrm{i}-z}\right) = \dfrac{-2xy(1-y) + x(x^2-y^2)}{x^2+(1-y)^2}$
- ☐ $\mathrm{Re}\left(\dfrac{z^2}{\mathrm{i}-z}\right) = \dfrac{-2xy(1+y) - x(x^2-y^2)}{x^2+(1+y)^2}$
- ■ $\mathrm{Re}\left(\dfrac{z^2}{\mathrm{i}-z}\right) = \dfrac{2xy(1-y) - x(x^2-y^2)}{x^2+(1-y)^2}$
- ☐ $\mathrm{Re}\left(\dfrac{z^2}{\mathrm{i}-z}\right) = \dfrac{2xy(1+y) + x(x^2-y^2)}{x^2+(1+y)^2}$

Question 8 : The series $\displaystyle\sum_{n=0}^{+\infty} \frac{2^n}{3^n+n}$

- ☐ diverges and the series $\displaystyle\sum_{n=0}^{+\infty} (-1)^n \frac{2^n}{3^n+n}$ converges
- ☐ converges and $\displaystyle\sum_{n=0}^{+\infty} \frac{2^n}{3^n+n} > 6$
- ■ converges and $\displaystyle\sum_{n=0}^{+\infty} \frac{2^n}{3^n+n} < 3$
- ☐ diverges and the series $\displaystyle\sum_{n=0}^{+\infty} (-1)^n \frac{2^n}{3^n+n}$ diverges

Second part, true/false questions

For each question, cross the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 9 : Let $x_0 \in \mathbb{R}$ and let (x_n) be the sequence defined recursivey by $x_{n+1} = \frac{1}{4}x_n + 1$ for $n \in \mathbb{N}$. Then, for all choices of x_0, the sequence (x_n) converges.

■ TRUE   □ FALSE

Question 10 : Let (a_n) be a sequence of real numbers $a_n \geq 0$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{+\infty} a_n$ converges, then the series $\sum_{n=0}^{+\infty} a_n^2$ converges.

■ TRUE   □ FALSE

Question 11 : Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function and $g : \mathbb{R} \to \mathbb{R}$ be a strictly decreasing function. Then the composition $f \circ g : \mathbb{R} \to \mathbb{R}$ is strictly decreasing.

■ TRUE   □ FALSE

Question 12 : Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Then the image of f is an open interval.

□ TRUE   ■ FALSE

Question 13 : Let (a_n) and (b_n) be two sequences of positive real numbers, such that $0 < a_n \leq b_n$ for all $n \in \mathbb{N}$. If the sequence (b_n) converges, then the sequence (a_n) converges.

□ TRUE   ■ FALSE

Question 14 : The function $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + e^x$ is bijective.

■ TRUE   □ FALSE