Analysis I (English)
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Analysis 1 - Exercise Set 9

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

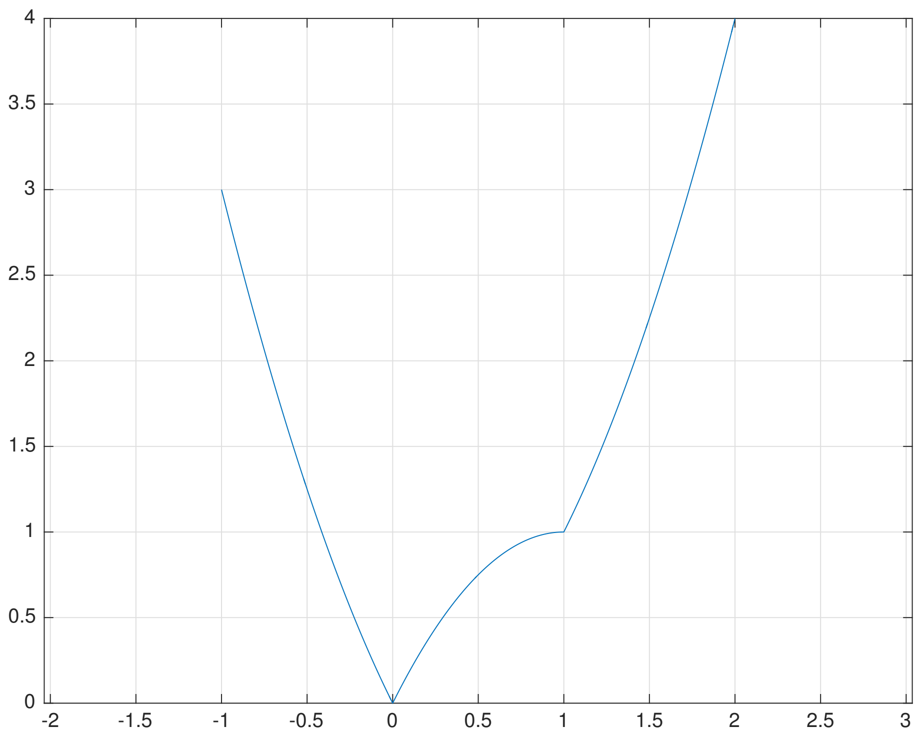
1. Find the local and global maximum/minimum of the function $f(x) = |x^2 - x| + |x|$, by sketching the graph of the function.

Solution:

We can rewrite the function as

$$f = \begin{cases} x^2 & \text{if} \quad x \geq 1 \\ -x^2 + 2x & \text{if} \quad 0 \leq x < 1 \\ x^2 - 2x & \text{if} \quad x < 0 \end{cases}$$

If we sketch the graph of this function we have



So the function attains its global minimum at $x = 0$ and has no global maximum.

2. Compute the following limits if they exist.

   (a) $\lim_{x\to 1} \frac{x^2-x}{x^2-2x+1}$

   (b) $\lim_{x\to +\infty} \left(\sqrt[3]{x+1} - \sqrt[3]{x}\right)$

   (c) $\lim_{x\to 0} \frac{(-1)^{[x]}}{\sin(x)^3} + \frac{1}{\sin(x)^2}$

Solution:

(a) We have

$$\frac{x^2-x}{x^2-2x+1} = \frac{x(x-1)}{(x-1)^2} = \frac{x}{x-1}$$

So the limit from the left is $-\infty$, and from the right is $+\infty$. We conclude that the limit does not exist.

(b) We use the formula $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ for $a = \sqrt[3]{x+1}$ and $b = \sqrt[3]{x}$ to obtain,

$$\lim_{x\to +\infty} \left(\sqrt[3]{x+1} - \sqrt[3]{x}\right) = \lim_{x\to +\infty} \frac{\left((x+1)^{\frac{1}{3}} - x^{\frac{1}{3}}\right)\left((x+1)^{\frac{2}{3}} + (x+1)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}{(x+1)^{\frac{2}{3}} + (x+1)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}$$

$$= \lim_{x\to +\infty} \frac{1}{(x+1)^{\frac{2}{3}} + (x+1)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}} = 0\,.$$

(c) We can assume that $x \in]-1, 1[$. So $[x] = 0$ for all $x \in]-1, 1[$. We compute the right and left limits.

$$\lim_{x\to 0^+} \frac{(-1)^{[x]}}{\sin(x)^3} + \frac{1}{\sin(x)^2} = \lim_{x\to 0^+} \frac{1}{\sin(x)^2}\left(\frac{(-1)^{[x]}}{\sin(x)} + 1\right) = \lim_{x\to 0^+} \frac{1}{\sin(x)^2}\left(\frac{1}{\sin(x)} + 1\right) = +\infty$$

because $\lim_{x\to 0^+} \sin(x) = 0$ and $\frac{1}{\sin(x)} \geq 0$ for $x \in]0, 1[$.

$$\lim_{x\to 0^-} \frac{(-1)^{[x]}}{\sin(x)^3} + \frac{1}{\sin(x)^2} = \lim_{x\to 0^-} \frac{1}{\sin(x)^2}\left(\frac{(-1)^{[x]}}{\sin(x)} + 1\right) = \lim_{x\to 0^-} \frac{1}{\sin(x)^2}\left(\frac{1}{\sin(x)} + 1\right) = -\infty$$

because $\frac{1}{\sin(x)} \leq 0$ for $x \in]-1, 0[$. So the limit does not exist, as left limit and right limit do not agree.

3. Consider the function

$$f(x) = \frac{x(x-1)\tan(x-1)}{x^3-3x+2},$$

whose domain is $\mathbb{R} \setminus \{-2, 1\}$.

   (a) Study its continuity at $x_0 = 0$.

   (b) Find, if it exists, a continuous extension of the function f in $x_0 = 1$, or otherwise show that f cannot have a continuous extension at $x_0 = 1$.

Solution:

(a) Yes, the function is continuous at x_0. We will use the theorems about composition, product, ratio, etc., of continuous functions. The function $x-1$ is continuous everywhere, as it is a polynomial. Thus, since the function tan is continuous everywhere, then so is the composition $\tan(x-1)$. Now, $x(x-1)$ is continuous everywhere, as it is a polynomial; then, as the product of two continuous functions is continuous, then $x(x-1)\tan(x-1)$ is continuous everywhere.

Now, x^3-3x+2 is continuous everywhere, as it is a polynomial. To conclude, we will use the fact that the ratio of two continuous functions is continuous, as long as the denominator is not 0. Thus, it suffices to check that x^3-3x+2 does not vanish at 0; but this is the case, as its value at $x_0=0$ is 2.

(b) If the limit $\lim\limits_{x\to x_0} f(x)$ exists in $\mathbb{R}$ then we can choose it as value a to extend f at x_0. We compute the limit of f at $x_0=1$. We can write the denominator of f as $x^3-3x+2=(x-1)^2(x+2)$ so we get

$$\begin{aligned}\lim_{x\to1} f(x) &= \lim_{x\to1}\frac{x(x-1)\tan(x-1)}{x^3-3x+2} = \lim_{x\to1}\frac{x(x-1)\tan(x-1)}{(x-1)^2(x+2)}\\ &= \lim_{x\to1}\left(\frac{x}{x+2}\cdot\frac{\tan(x-1)}{x-1}\right) = \lim_{x\to1}\frac{x}{x+2}\cdot\lim_{x\to1}\frac{\tan(x-1)}{x-1}\\ &= \frac{1}{3}\cdot\lim_{x\to1}\left(\frac{\sin(x-1)}{(x-1)}\cdot\frac{1}{\cos(x-1)}\right) = \frac{1}{3}\cdot 1 = \frac{1}{3}\end{aligned}$$

(*Attention:* The decomposition of the product of the two limits in the second line exists because both limits exist.)

So the continuous extension of f is

$$\hat{f}_1\colon \mathbb{R}\setminus\{-2\}\longrightarrow\mathbb{R},\qquad \hat{f}_1(x)=\begin{cases}\dfrac{x(x-1)\tan(x-1)}{x^3-3x+2}, & x\neq 1,-2\\ \frac{1}{3}\ , & x=1.\end{cases}$$

4. (a) Prove or disprove that a function is continuous if and only if it is uniformly continuous.
   (b) Prove or disprove that $f\colon\mathbb{R}\to\mathbb{R}$, $f(x)=\sin(x)$ is a uniformly continuous function.
   (c) Show that the function $f\colon]0,b[\to\mathbb{R}$ defined by $f(x)=x^2$ is continuous and also uniformly continuous for $b<+\infty$. Show that f is not uniformly continuous when $b=+\infty$.

Solution:

(a) This is not true. The function $f\colon]0,+\infty[\mapsto\mathbb{R}$ defined by $f(x)=\frac{1}{x}$ is continuous, but not uniformly continuous.

(b) Let $\varepsilon>0$. For $x,y\in\mathbb{R}$, we have

$$|\sin(x)-\sin(y)| = 2\left|\cos\left(\frac{x+y}{2}\right)\right|\cdot\left|\sin\left(\frac{x-y}{2}\right)\right| \le 2\left|\sin\left(\frac{x-y}{2}\right)\right| \le 2\left|\frac{x-y}{2}\right| = |x-y|.$$

So if $|x-y|\le\delta$ with $\delta=\varepsilon$, then $|\sin(x)-\sin(y)|\le\varepsilon$. Thus $\sin(.)$ is uniformly continuous.

(c) It is continuous because,

$$|f(x)-f(x_0)| = |x^2-x_0^2| = |x+x_0||x-x_0| < 2b|x-x_0|$$

So it is enough to take $\delta = \frac{\epsilon}{2b}$.

It is uniformly continuous because,

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 2b|x - y|$$

So it is enough to take $\delta = \frac{\epsilon}{2b}$.

We want to show that if the domain of f is $(0, \infty)$, then f is not uniformly continuous. It means that we have to find an $\varepsilon > 0$ such that for any $\delta > 0$ we can find $x_0, y_0 \in (0, \infty)$ that satisfy $|x_0 - y_0| < \delta$ but $|f(x_0) - f(y_0)| > \varepsilon$. Let's take $\varepsilon = 1$, $x_0 = N$ some arbitrary integer and $y_0 = N + \delta/2$ for some arbitrary $\delta > 0$. Then

$$|f(x_0) - f(y_0)| = |x_0^2 - y_0^2| = |x_0 - y_0||x_0 + y_0| = |\frac{\delta}{2}||2N + \delta/2|$$

Now if f is uniformly continuous then we require $|f(x_0) - f(y_0)| < 1$ and consequently $|\frac{\delta}{2}||2N + \delta/2| < 1$. But this is a contradiction because for a fixed δ we can pick N to be arbitrary large so that the inequality is violated. So f cannot be uniformly continuous.

5. Let I be an interval, $f\colon I \to \mathbb{R}$ be a continuous function and $f(I)$ the image of I by f. Say if the following statement are true or false.
   (a) $f(I)$ is an interval (where here we also admit the degenerate case $f(I) = [m, m] = \{m\}$).
   (b) If I is a bounded and closed interval, then $f(I)$ is a bounded and closed interval (where here we also admit the degenerate case $f(I) = [m, m] = \{m\}$).
   (c) If I is open, then $f(I)$ is an open interval.
   (d) If $I = [a, b[$ with $a, b \in \mathbb{R}$, $a < b$, then f attains its maximum and minimum in I. That is, there exists $m, M \in R(f)$ such that $R(f) = [m, M]$.

Solution:

(a) True. It is a consequence of the intermediate value theorem.

(b) True. It is a consequence of the intermediate value theorem.

(c) False. Take, for example, the function $f\colon]-1, 1[\to \mathbb{R}$ defined by $f(x) = \frac{1}{-x^2+1}$. Then I is open but $f(I) = [1, +\infty[$ is not open.

(d) False. For example take the function $f\colon [-1, 0[\to \mathbb{R}$ defined by $f(x) = \frac{1}{x}\sin\big(\frac{1}{x}\big)$. Then f neither have a minimum nor a maximum on I because for all $n \in \mathbb{N}^*$, we have $-\big(2\pi n \pm \frac{\pi}{2}\big)^{-1} \in I$ but

$$\begin{aligned} f\left(-\frac{1}{2\pi n + \frac{\pi}{2}}\right) &= -\left(2\pi n + \frac{\pi}{2}\right)\sin\left(-\left(2\pi n + \frac{\pi}{2}\right)\right) = \left(2\pi n + \frac{\pi}{2}\right)\sin\left(2\pi n + \frac{\pi}{2}\right) \\ &= \left(2\pi n + \frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 2\pi n + \frac{\pi}{2} > n \end{aligned}$$

and

$$\begin{aligned} f\left(-\frac{1}{2\pi n - \frac{\pi}{2}}\right) &= -\left(2\pi n - \frac{\pi}{2}\right)\sin\left(-\left(2\pi n - \frac{\pi}{2}\right)\right) = \left(2\pi n - \frac{\pi}{2}\right)\sin\left(2\pi n - \frac{\pi}{2}\right) \\ &= \left(2\pi n - \frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) = -2\pi n + \frac{\pi}{2} < -n. \end{aligned}$$

6. Find, if it exists, continuous extension of the function f :$]0,1] \to \mathbb{R}$ given by $f(x) = \frac{\tan(\sqrt{1+x}-1)}{x^{3/2}}$ at $x_0 = 0$, or otherwise show that f cannot have a continuous extension at x_0. (*Note: you have to care just about the limit from the right, that is:* $x \to 0^+$)

Solution:
We check if the limit of f exists as $x \to 0^+$. We have

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\tan(\sqrt{1+x}-1)}{x^{3/2}} &= \lim_{x\to 0^+} \frac{\frac{\sin(\sqrt{1+x}-1)}{\cos(\sqrt{1+x}-1)}}{x^{3/2}} \\
&= \lim_{x\to 0^+} \frac{\sin(\sqrt{1+x}-1)}{x^{3/2}} \cdot \frac{1}{\cos(\sqrt{1+x}-1)} \\
&= \lim_{x\to 0^+} \frac{\sin(\sqrt{1+x}-1)}{x^{3/2}} \cdot \frac{\sqrt{1+x}-1}{\sqrt{1+x}-1} \cdot \frac{1}{\cos(\sqrt{1+x}-1)} \\
&= \lim_{x\to 0^+} \frac{\sqrt{1+x}-1}{x^{3/2}} \cdot \frac{\sin(\sqrt{1+x}-1)}{\sqrt{1+x}-1} \cdot \frac{1}{\cos(\sqrt{1+x}-1)}
\end{aligned}$$

Note that the limit of the second fraction is finite due to the sandwich theorem ($\sqrt{1-y^2} \leq \frac{\sin y}{y} \leq 1$) so $\lim_{x\to 0} \frac{\sin(\sqrt{1+x}-1)}{\sqrt{1+x}-1} = 1$. The limit of the last fraction is also finite as $\lim_{x\to 0} \frac{1}{\cos(\sqrt{1+x}-1)} = 1$. For the limit of the first fraction we have:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sqrt{1+x}-1}{x^{3/2}} &= \lim_{x\to 0^+} \frac{\sqrt{1+x}-1}{x^{3/2}} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \\
&= \lim_{x\to 0^+} \frac{x}{x^{3/2}} \cdot \frac{1}{\sqrt{1+x}+1} \\
&= \lim_{x\to 0^+} \frac{1}{x^{1/2}} \cdot \frac{1}{\sqrt{1+x}+1} = +\infty
\end{aligned}$$

So the function $\frac{\tan(\sqrt{1+x}-1)}{x^{3/2}}$ does not have a limit in $\mathbb{R}$ as $x \to 0^+$, hence we cannot have a continuous extension of this function.

7. Use the intermediate value theorem to show that the following equations have at least one solution in $\mathbb{R}$:

(a) $e^{x-1} = x + 1$

(b) $x^2 - \dfrac{1}{x} = 1$

Solution:

(a) To use the intermediate value theorem, we must define a continuous function starting from the given equation. So, let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x-1} - x - 1$. Then f is continuous in $\mathbb{R}$ since it is the combination of continuous functions and since $e = 2.7182\ldots$, we have $f(2) = e - 3 < 0$ and $f(3) = e^2 - 4 > 0$. By the intermediate value theorem, there exist $x_0 \in [2,3]$ such that $f(x_0) = 0$.

Note also that this equation also admits another root. In fact, we have $f(0) = \frac{1}{e} - 1 < 0$ and $f(-1) = \frac{1}{e^2} > 0$ and by the intermediate value theorem there exists $x_0 \in [-1,0]$ such that $f(x_0) = 0$.

(b) To use the intermediate value theorem, we must define a continuous function starting from the given equation. Since the given equation is not defined at $x = 0$, we need to define the function f on $]-\infty, 0[$ and on $]0, \infty[$

If $x < 0$, we have $x^2 - \frac{1}{x} = x^2 + \frac{1}{|x|} > 1$ because one of the two terms is always ≥ 1 so the equation does not take any roots. So we define $f\colon]0, \infty[\to \mathbb{R}$, $f(x) = x^2 - \frac{1}{x} - 1$. This function is continuous (sum of continuous functions) and we have $f(1) = -1 < 0$ and $f(2) > 0$. By the intermediate value theorem, there exists $x_0 \in [1, 2]$ such that $f(x_0) = 0$.

8. State if the following functions are continuous and differentiable at $x = 0$.

(a) $|\sin(x)|$

(b) $|x^3|$

Solution:

(a) The function $\sin(x)$ is continuous and differentiable at $x = 0$. The function $|.|$ is continuous, so $|\sin(x)|$ is continuous, as it is the composition of two continuous functions. If we look at the graph of $|\sin(x)|$ we see that there are two tangent lines at $x = 0$. Hence, we expect that the function is not differentiable at $x = 0$. Let's compute the derivative, if it exists. We distinguish right and left limits

$$\lim_{x\to 0^+} \frac{|\sin(x)|}{x} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1, \qquad \lim_{x\to 0^-} \frac{|\sin(x)|}{x} = \lim_{x\to 0^-} \frac{-\sin(x)}{x} = -1.$$

So the limit $\lim_{x\to 0} \frac{|\sin(x)|}{x}$ does not exist and $|\sin(x)|$ is not differentiable at $x = 0$.

(b) The function x^3 is continuous and differentiable everywhere because it is a polynomial. Since $|.|$ is continuous, then $|x^3|$ is continuous everywhere because of composition of continuous functions. The derivative at $x = 0$, if it exists, is the limit $\lim_{x\to 0} \frac{|x^3|}{x}$. We distinguish the right and left limit.

$$\lim_{x\to 0^+} \frac{|x^3|}{x} = \lim_{x\to 0^+} \frac{x^3}{x} = 0, \qquad \lim_{x\to 0^-} \frac{|x^3|}{x} = \lim_{x\to 0^-} \frac{-x^3}{x} = 0.$$

Hence the limit exists and $|x^3|$ is differentiable at $x = 0$.

9. Check if the following functions are uniformly continuous

(a) $\sqrt{x}$ with domain $[0, +\infty)$

(b) x^3 with domain $[0, \pi]$

(c) x^3 with domain $\mathbb{R}$

Solution:

(a) $\sqrt{x}$ with domain $[0, +\infty)$ is uniformly continuous. We observe that since $\sqrt{x}, \sqrt{y} \geq 0$, we have $|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} + \sqrt{y}|$. Then

$$|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}| \cdot |\sqrt{x} + \sqrt{y}| = |x - y|$$

so we can take $\delta = \varepsilon^2$ in the definition of uniform continuity.

(b) x^3 with domain $[0, \pi]$ is uniformly continuous. We have

$$|y^3 - x^3| = |y - x| \cdot |y^2 + xy + x^2| \leq |y - x| 3\pi^2$$

So, for every $\varepsilon > 0$, if we choose $\delta < 1$ and

$$\delta < \frac{\varepsilon}{3\pi^2}$$

then if $|x - y| < \delta$ we get that

$$|y^3 - x^3| \leq \varepsilon$$

(c) x^3 with domain $\mathbb{R}$ is not uniformly continuous. To get a counterexample, take $\varepsilon = 1$. For every $\delta > 0$, take $x \geq \frac{2}{3\delta^2}$ and $y = x + \delta$, and we have

$$|y^3 - x^3| = |(x + \delta)^3 - x^3| = |2x^2\delta + 3x\delta^2| > 3x\delta^2 \geq 2 > \varepsilon.$$

10. Let f and g be two continuous functions in $[a, b]$, such that $f(a) > g(a)$ and $f(b) < g(b)$. Show that there is $c \in]a, b[$ such that $f(c) = g(c)$. (*Hint: use the function $h = f - g$ and the intermediate value theorem.*)

Solution:

Define the function $h(x) = f(x) - g(x)$. Since f and g are continuous functions in $[a, b]$ then h is also a continuous function on $[a, b]$. Also we have that $h(a) = f(a) - g(a) > 0$ since $f(a) > g(a)$ and $h(b) = f(b) - g(b) < 0$ since $f(b) < g(b)$. So h satisfies the intermidiate value theorem so there exists a $c \in]a, b[$ such that $h(c) = 0$. But $h(c) = 0$ implies that $f(c) - g(c) = 0$ meaning that $f(c) = g(c)$.

11. Find the inverse of the following functions if they exist. Give the domain of both functions.

(a) $f(x) = \sqrt{(2x + 4)^3 - 7}$

(b) $f(x) = \frac{2x+3}{3x+5}$

(c) $f(x) = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}$

Solution:

(a) To find the domain of f we notice that only positive values are accepted under the square root sign:

$$(2x + 4)^3 - 7 \geq 0 \Longrightarrow x \geq -2 + \frac{\sqrt[3]{7}}{2}$$

So $D_f = [-2 + \frac{\sqrt[3]{7}}{2}, +\infty[$. We observe that f is injective because it is a composition of injective functions: $\sqrt{x}$, $x - 7$, x^3 and $2x + 4$ are all injective. Hence, an inverse function exists. To find the inverse function we have:

$$y = \sqrt{(2x + 4)^3 - 7} \Longrightarrow \frac{\sqrt[3]{y^2 + 7}}{2} - 2 = x$$

So the inverse function is $f^{-1}(x) = \frac{\sqrt[3]{x^2+7}}{2} - 2$ and its domain is the entire real numbers $D_{f^{-1}} = \mathbb{R}$.

(b) To find the domain of f we notice that the denominator cannot become zero so $D_f = \mathbb{R}\backslash\{-5/3\}$. To find the inverse function we have:

$$y = \frac{2x+3}{3x+5} \Longrightarrow 3yx + 5y = 2x + 3 \Longrightarrow x(3y-2) = 3 - 5y \Longrightarrow x = \frac{3-5y}{3y-2}$$

where we assumed that $x \neq -5/3$ and $y \neq 2/3$. So x is uniquely determined by y, in particular the function f is injective. So the inverse function exists and is given by $f^{-1}(x) = \frac{3-5x}{3x-2}$ with the domain $D_{f^{-1}} = \mathbb{R}\backslash\{2/3\}$.

(c) Note that $2\sin x \cos x = \sin(2x)$. Since the denominator cannot become zero, we have that $x \neq \frac{k\pi}{2}$ where $k \in \mathbb{Z}$. So the domain of f is $\mathbb{R}\backslash\{\frac{k\pi}{2} | k \in \mathbb{Z}\}$. We have

$$f(x) = \frac{\cos^2 x - \sin^2 x}{2\sin x \cos x} = \frac{\cos(2x)}{\sin(2x)} = \cot(2x)$$

We observe that the function f is not injective because it is periodic. Hence the inverse function does not exist. But $g := f|_{]0,\frac{\pi}{2}[} :]0, \frac{\pi}{2}[\to \mathbb{R}$ is injective with inverse function $g^{-1}(x) = \frac{1}{2}\operatorname{arccot} x$. The domain of the inverse function is $D_{g^{-1}} = \mathbb{R}$.

12. Let I be an interval, $f\colon I \to \mathbb{R}$ be a continuous function and $f(I)$ the image of I by f. Say if the following statement are true or false.

(a) If I is bounded, then $f(I)$ is bounded.

(b) If $I = [a, \infty[$ with $a \in \mathbb{R}$, then f attains its maximum and minimum in I.

(c) If f is strictly increasing and I is open, then $f(I)$ is open.

Solution:

(a) False. Take for example the function $f\colon]0,1[\to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Then I is bounded but $f(I) =]1, \infty[$ is not bounded.

(b) False. Take for example the function defined by $f(x) = (x-a)\sin(x-a)$. It neither have a minimum nor a maximum on I because for all $n \in \mathbb{N}$, we have $f\big(a + \frac{\pi}{2} + 2\pi n\big) = \frac{\pi}{2} + 2\pi n > n$ and $f\big(a - \frac{\pi}{2} + 2\pi n\big) = \frac{\pi}{2} - 2\pi n < -n$.

(c) True. Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ such that $I =]a, b[$. Since the function is strictly increasing, we have

$$A := \lim_{x \to a^+} f(x) = \inf\{f(x) : x \in]a,b[\} < \sup\{f(x) : x \in]a,b[\} = \lim_{x \to b^-} f(x) =: B$$

and A, B defined above belong to $\mathbb{R} \cup \{\pm\infty\}$. In particular, $f(I) \subseteq [A, B]$. We observe that if $A \in f(I)$, then there exists $x \in]a, b[$ such that $f(x) = A$. Since f is strictly increasing, then $x \leq a$, which is impossible because $x > a$. Hence $A \notin f(I)$. Similarly we prove that $B \notin f(I)$. So we conclude that $f(I) \subseteq]A, B[$. Now we observe that by definition of limits we can find sequences (x_n), (y_n) contained in $]a, b[$ such that (x_n) converges to a and (y_n) converges to b and $x_n \leq y_n$ for all $n \in \mathbb{N}$, so that $(f(x_n))$ converges to A and $(f(y_n))$ converges to B, and hence $]A, B[= \cup_{n\in\mathbb{N}}[f(x_n), f(y_n)]$. By the intermediate value theorem, we have $[f(x_n), f(y_n)] \subseteq f(I)$ for all $n \in \mathbb{N}$. Then

$$]A, B[= \cup_{n\in\mathbb{N}}[f(x_n), f(y_n)] \subseteq f(I).$$

We proved that $f(I)$ is the open interval $]A, B[$.

13. Find, if it exists, continuous extension of the function $f :]2, \infty[\to \mathbb{R}$ give by $f(x) = \frac{\sqrt{x}-\sqrt{2}+\sqrt{x-2}}{\sqrt{x^2-4}}$ at $x_0 = 2$, or otherwise show that f cannot have a continuous extension at x_0.

Solution:

We check if the limit $\lim_{x\to 2^+} f(x)$ exists. We have:

$$\begin{aligned}
\lim_{x\to 2^+} \frac{\sqrt{x}-\sqrt{2}+\sqrt{x-2}}{\sqrt{x^2-4}} &= \lim_{x\to 2^+} \left(\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x-2}\cdot\sqrt{x+2}} + \frac{\sqrt{x-2}}{\sqrt{x-2}\cdot\sqrt{x+2}} \right) \\
&= \lim_{x\to 2^+} \left(\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x-2}\cdot\sqrt{x+2}} \cdot \frac{\sqrt{x-2}}{\sqrt{x-2}} \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}} + \frac{1}{\sqrt{x+2}} \right) \\
&= \lim_{x\to 2^+} \left(\frac{(x-2)}{(x-2)\cdot\sqrt{x+2}} \cdot \frac{\sqrt{x-2}}{\sqrt{x}+\sqrt{2}} + \frac{1}{\sqrt{x+2}} \right) \\
&= 0 + \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

So we can extend the function f continuously to the interval $[2, \infty[$. We define the new function as:

$$\hat{f}(x) = \begin{cases} f(x), & x > 2 \\ 1/2, & x = 2 \end{cases}$$

which is continuous on $[2, \infty[$.

14. **The Bisection Algorithm:** Using the intermediate value theorem and successive bisection of the interval $[0, 1]$, find an interval of the length $L \leq \frac{1}{8}$ that contains a solution of the equation

$$x^3 + x - 1 = 0.$$

Solution:

We search for a root x_0 of the function $f(x) = x^3 + x - 1$, i.e. x_0 that satisfies $f(x_0) = 0$. The bisection algorithm is an iterative routine that successively restricts the interval that contains the root x_0.

The steps of the algorithm is given below, where L is the length of the interval where we can find a root x_0.

$$\begin{aligned}
&f(0) = -1 < 0 \quad \text{et} \quad f(1) = 1 > 0 && \Longrightarrow && x_0 \in]0, 1[, \quad L = 1 \\
&f\left(\tfrac{1}{2}\right) = -\tfrac{3}{8} < 0 && \Longrightarrow && x_0 \in \left]\tfrac{1}{2}, 1\right[, \quad L = \tfrac{1}{2} \\
&f\left(\tfrac{3}{4}\right) = 0.172 > 0 && \Longrightarrow && x_0 \in \left]\tfrac{1}{2}, \tfrac{3}{4}\right[, \quad L = \tfrac{1}{4} \\
&f\left(\tfrac{5}{8}\right) = -0.130 < 0 && \Longrightarrow && x_0 \in \left]\tfrac{5}{8}, \tfrac{3}{4}\right[, \quad L = \tfrac{1}{8}
\end{aligned}$$

So $x_0 \in \left]\frac{5}{8}, \frac{3}{4}\right[=]0.625, 0.75[$. The exact value of the root is $x_0 = 0.6823\ldots$ which is indeed inside the obtained interval.

15. Let the function $f\colon [0,\infty) \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \dfrac{3x^2 - 10x + 3}{x^2 - 2x - 3}\,, & x > 3 \\ \alpha\,, & x = 3 \\ \beta x - 4\,, & x < 3 \end{cases}$$

Find $\alpha, \beta \in \mathbb{R}$ such that the function is continuous at $x = 3$.

Solution:
For the function to be continuous we require $\lim_{x\to 3^-} f = \lim_{x\to 3^+} f = f(3)$We first compute the right limit at $x = 3$:

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{3x^2 - 10x + 3}{x^2 - 2x - 3} = \lim_{x\to 3^+} \frac{(x-3)(3x-1)}{(x-3)(x+1)} = 2$$

This implies that $f(3) = \alpha = 2$. Also for the limit from the left we have:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \beta x - 4 = 3\beta - 4$$

So $3\beta - 4 = 2$, and we get $\beta = 2$.

16. Show that if $f(x)$ is continuous on $[-1, 1]$ and $f(-1) = f(1)$, then there exists $\delta \in [0, 1]$ such that $f(\delta) = f(\delta - 1)$.

Solution:
First, consider the function $g(x) = f(x-1) - f(x)$, which is continuous on $[0, 1]$. Now we know the following:

$$g(0) = f(-1) - f(0)$$

and

$$g(1) = f(0) - f(1)$$

But we know that $f(-1) = f(1)$, so we can manipulate the second equation to get

$$g(1) = f(0) - f(1) = f(0) - f(-1) = -g(0)$$

- If $g(0) = 0$, then $f(-1) = f(0)$ by the first equation. Hence, $\delta = 0 \in [0, 1]$.
- If $g(0) > 0$, then $g(1) < 0$ so by the intermediate value theorem there exists a $\delta \in [0, 1]$ such that $g(\delta) = 0$, meaning $f(\delta - 1) = f(\delta)$.
- If $g(0) < 0$, then $g(1) > 0$ so by the intermediate value theorem there exists a $\delta \in [0, 1]$ such that $g(\delta) = 0$, meaning $f(\delta - 1) = f(\delta)$.

17. Find, if it exists, continuous extension of the function $f : [-\pi/4, 0[\cup]0, \pi/4] \to \mathbb{R}$ given by $f(x) = \frac{1-\cos x}{\tan^2 x}$ at $x_0 = 0$, or otherwise show that f cannot have a continuous extension at x_0.

Solution:

We check if the limit of f exists as $x \to 0$. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{\tan^2 x} &= \lim_{x\to 0} \frac{1-\cos x}{\frac{\sin^2 x}{\cos^2 x}} \cdot \frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{\sin^2 x} \cdot \frac{\cos^2 x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{\sin^2 x} \cdot \frac{\cos^2 x}{1+\cos x} \\
&= \frac{1}{2}
\end{aligned}$$

So we can extend the function f continuously to the interval $[-\pi/4, \pi/4]$. We define the new function as:

$$\hat{f}(x) = \begin{cases} f(x), & x \neq 0 \\ 1/2, & x = 0 \end{cases}$$

which is continuous on $[-\pi/4, \pi/4]$.

18. Let us define the functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

(a) Find domain and range for each of the 3 functions.

(b) Show that

$$\cosh(x)^2 - \sinh(x)^2 = 1.$$

(c) Find a suitable domain, for each of the 3 functions, over which the function is invertible.

(d) Compute

$$\begin{aligned}
&\lim_{x\to+\infty} \cosh(x), \quad \lim_{x\to-\infty} \cosh(x), \\
&\lim_{x\to+\infty} \sinh(x), \quad \lim_{x\to-\infty} \sinh(x), \\
&\lim_{x\to+\infty} \tanh(x), \quad \lim_{x\to-\infty} \tanh(x).
\end{aligned}$$

Solution:

(a) As e^x, e^{-x} are defined over all of the real numbers, that is, $Dom(e^x) = Dom(e^{-x}) = \mathbb{R}$, then the same holds true for $\sinh(x), \cosh(x)$. Moreover, $\cosh(x) > 0$, $\forall x \in \mathbb{R}$, hence also $Dom(\tanh(x)) = \mathbb{R}$.
Let us examine the ranges. Let us notice that $\cosh(x)$ is even, while $\sinh(x)$ is odd, and $\tanh(x)$ is odd, as well.
$\sinh(x)$ is strictly increasing on the interval $[0, +\infty[$: in fact, taking $s > t > 0$, then $e^s > e^t$ (or, equivalently, $e^s - e^t > 0$), $e^{-s} < e^{-t}$ (or, equivalently, $e^{-s} - e^{-t} < 0$), so that

$$\sinh(s) > \sinh(t) \iff e^s - e^{-s} > e^t - e^{-t} \iff e^s - e^t > e^{-s} - e^{-t}.$$

But finally, $e^{-s} - e^{-t} < 0$, while $e^s - e^t > 0$, which show that the last inequality above indeed holds.

(b) By the definitions,

$$\cosh(x)^2 = \frac{(e^x + e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4} = \frac{e^{2x} + e^{-2x} + 2e^{x-x}}{4} = \frac{e^{2x} + e^{-2x} + 2}{4},$$
$$\sinh(x)^2 = \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4} = \frac{e^{2x} + e^{-2x} - 2e^{x-x}}{4} = \frac{e^2 x + e^{-2x} - 2}{4},$$
$$\implies \cosh(x)^2 - \sinh(x)^2 = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = \frac{4}{4} = 1.$$

(c) As $\cosh(x) = \sqrt{1 + \sinh(x)^2}$ and over the positive real numbers the functions $f(x) = \sinh(x)$, $g(x) = 1 + x^2$, $h(x) = \sqrt{x}$ are all strictly increasing and $\cosh(x) = h(g(f(x)))$, then $cosh(x)$ is strictly increasing on $[0, +\infty[$. As $\cosh(x)$ is even, this is the largest interval over which the function is injective (hence, invertible when we take the arrival set to coincide with $R(\cosh(x)) = [1, +\infty[$, since $\cosh(0) = 1$ and $\lim_{x\to+\infty} \cosh(x) = +\infty$, and, again, $\cosh(x)$ is even).
As $\sinh(x)$ is odd, and strictly increasing on $[0, +\infty[$, then it is strictly increasing over all of $\mathbb{R}$ and it is therefore invertible on $\mathbb{R}$ and $R(f) = \mathbb{R}$, since $\sinh(0) = 0$ and $\lim_{x\to+\infty} \sinh(x) = +\infty$, and, again, $\sinh(x)$ is odd.

(d) Let us note that $\cosh(x) \geq |x|$, $\forall x \in \mathbb{R}$. Hence, the squeeze theorem implies immedaitely that

$$\lim_{x\to+\infty} \cosh(x) = +\infty = \lim_{x\to-\infty} \cosh(x).$$

Since $\lim_{x\to+\infty} e^x = +\infty$, then $\lim_{x\to+\infty} e^{-x} = 0$ and

$$\lim_{x\to+\infty} \frac{\sinh(x)}{e^x} = \lim_{x\to+\infty} \frac{\frac{e^x - e^{-x}}{2}}{e^x} = \lim_{x\to+\infty} \frac{\frac{e^x}{e^x} - \frac{e^{-x}}{e^x}}{2} = \frac{1}{2}.$$

Hence $\sinh(x) \geq \frac{1}{3}e^x$[1] for $x \gg 0$, hence

$$\lim_{x\to+\infty} \sinh(x) = +\infty.$$

As $\sinh(x)$ is odd, then

$$\lim_{x\to-\infty} \sinh(x) = -\infty.$$

Finally,

$$\lim_{x\to+\infty} \tanh(x) = \lim_{x\to+\infty} \frac{\sinh(x)}{\cosh(x)} = \lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} =$$
$$\lim_{x\to+\infty} \frac{e^{-x} \cdot (e^x - e^{-x})}{e^{-x} \cdot (e^x + e^{-x})} = \lim_{x\to+\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1,$$

since $\lim_{x\to+\infty} e^{-x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$.
As $\tanh(x)$ is odd, then

$$\lim_{x\to-\infty} \tanh(x) = -1.$$