

Analysis 1 - Exercise Set 7

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Show that for every $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k=1}^n (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}} = 1 + \frac{(-1)^{n+1}}{2n+1}.$$

- (b) Use the previous part to compute $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}}$.

- (c) Is the series absolutely convergent?

Solution:

- (a) We prove it by induction on n . For $n = 1$ both sides are equal to $\frac{4}{3}$. Assume that $n > 1$ and that the equality holds for $n - 1$. Then

$$\begin{aligned} \sum_{k=1}^n \frac{(-1)^{k+1} k}{k^2 - \frac{1}{4}} &= 1 + \frac{(-1)^{(n-1)+1}}{2(n-1)+1} + \frac{(-1)^{n+1} n}{n^2 - \frac{1}{4}} = 1 + \frac{(-1)^n}{2n-1} + \frac{4(-1)^{n+1} n}{4n^2 - 1} \\ &= 1 + (-1)^n \left(\frac{1}{2n-1} - \frac{4n}{(2n-1)(2n+1)} \right) = 1 + (-1)^n \frac{2n+1-4n}{(2n-1)(2n+1)} \\ &= 1 + (-1)^{n+1} \frac{2n-1}{(2n-1)(2n+1)} = 1 + (-1)^{n+1} \frac{1}{2n+1} \end{aligned}$$

- (b)

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}} = \lim_{n \rightarrow \infty} 1 + \frac{(-1)^{n+1}}{2n+1} = 1.$$

- (c) We observe that $0 \leq \left| (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}} \right| = \frac{1}{k - \frac{1}{4k}}$ and $\frac{1}{k - \frac{1}{4k}} \geq \frac{1}{k}$ for all $k \geq 1$. Since the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k}{k^2 - \frac{1}{4}}$ does not converge absolutely by the comparison criterion (see week 6 worksheet).

2. Determine for which values of $a \in ((0, +\infty) \setminus \{1\})$, the series

$$\sum_{n=1}^{\infty} \frac{a^n}{a^{2n} - 1}$$

converges. (*Hint: use the Cauchy criterion.*)

Solution:

We observe that

$$b_n := \sqrt[n]{\left| \frac{a^n}{a^{2n} - 1} \right|} = \frac{a}{\sqrt[n]{|a^{2n} - 1|}} = \frac{1}{a \sqrt[n]{|1 - (a^{-1})^{2n}|}}$$

If $a < 1$, then $\frac{a}{\sqrt[n]{|a^{2n} - 1|}} = \frac{a}{\sqrt[n]{1 - a^{2n}}}$. Moreover, the sequence a^{2n} decreases, then the sequence $1 - a^{2n}$ increases. So

$$\sqrt[n]{1 - a} \leq \sqrt[n]{1 - a^{2n}} \leq 1$$

for all $n \geq 1$. Since $(\sqrt[n]{1 - a})$ converges with limit 1, the sequence $(\sqrt[n]{1 - a^{2n}})$ converges with limit 1 by squeeze theorem. Then (b_n) converges with limit a . Since $a < 1$ we conclude that the series converges by the Cauchy criterion.

If $a > 1$, then $a^{-1} < 1$ and $\frac{1}{a \sqrt[n]{|1 - (a^{-1})^{2n}|}} = \frac{a^{-1}}{\sqrt[n]{1 - (a^{-1})^{2n}}}$. Repeating the previous argument with a replaced by a^{-1} , we obtain that (b_n) converges with limit a^{-1} . Since $a^{-1} < 1$ we conclude that the series converges by the Cauchy criterion.

3. Compute $\lim_{n \rightarrow \infty} \sqrt[n]{n!}$

Solution:

First note that $n! \geq n(n-1)(n-2) \cdots \lfloor \frac{n}{2} \rfloor \geq \lfloor \frac{n}{2} \rfloor^{\lfloor \frac{n}{2} \rfloor}$. Thus, $\sqrt[n]{n!} \geq \lfloor \frac{n}{2} \rfloor^{\lfloor \frac{n}{2} \rfloor \frac{1}{n}}$. We can also see that $\lfloor \frac{n}{2} \rfloor^{\frac{1}{n}} \geq \frac{1}{10}$ for sufficiently large n . Thus, $\sqrt[n]{n!} \geq \lfloor \frac{n}{2} \rfloor^{\lfloor \frac{n}{2} \rfloor \frac{1}{n}} \geq \lfloor \frac{n}{2} \rfloor^{\frac{1}{10}}$ and the latter diverges to $+\infty$. Hence, $\lim_{n \rightarrow \infty} \sqrt[n]{n!} = +\infty$.

4. Let (x_n) be a sequence.

- Show that if $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = \rho$, with $\rho > 1$, then (x_n) is unbounded. In particular, it diverges.
- Show that if $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = \rho$, with $0 \leq \rho < 1$, then (x_n) converges to 0. In particular, it is bounded.
- Provide two sequences (y_n) and (z_n) with the following properties: (y_n) converges and $\lim_{n \rightarrow \infty} \sqrt[n]{|y_n|} = 1$, and (z_n) diverges to $+\infty$ and $\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = 1$

Solution:

- Note that we have $\lim_{n \rightarrow \infty} \frac{\sqrt[n]{|x_n|}}{\rho} = 1$.

Suppose that (x_n) is bounded, say $|x_n| \leq C \quad \forall n \in \mathbb{N}$ for some positive constant C . Thus,

$$1 = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{|x_n|}}{\rho} \leq \lim_{n \rightarrow \infty} \frac{\sqrt[n]{C}}{\rho} = \frac{1}{\rho} < 1$$

since $\rho > 1$. This is a contradiction and therefore (x_n) is unbounded and it must diverge.

(b) Note that we have $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = \rho$ with $0 \leq \rho < 1$.

Let $\epsilon = \frac{1-\rho}{2}$. Then, by definition of convergence, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$, $|\sqrt[n]{|x_n|} - \rho| \leq \frac{1-\rho}{2}$. In particular, for every $n \geq N$, we have $0 \leq \sqrt[n]{|x_n|} \leq \frac{1+\rho}{2}$. Also, notice that $\frac{1+\rho}{2} < 1$. Then, by raising the inequality to the n -th power, we have

$$0 \leq |x_n| \leq \left(\frac{1+\rho}{2}\right)^n.$$

In turn, we have

$$-\left(\frac{1+\rho}{2}\right)^n \leq x_n \leq \left(\frac{1+\rho}{2}\right)^n$$

for every $n \geq N$. Since $0 < \frac{1+\rho}{2} < 1$, its geometric sequence converges to 0. Then, by the squeeze theorem, also (x_n) converges to 0.

(c) Consider the constant sequence $y_n = 5$ for all n and the sequence $z_n = n$.

5. Let $(t_n) \subset \mathbb{R}^*$ be a sequence. Assume that $\lim_{n \rightarrow \infty} t_n = 0$. Show that $\lim_{n \rightarrow \infty} \frac{\sin(t_n)}{t_n} = 1$.
(Hint: recall that for $x \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} 0 \leq \sin(x) \leq x \leq \tan(x) &\Rightarrow 1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)} \Rightarrow \cos(x) \leq \frac{\sin(x)}{x} \leq 1 \\ &\Rightarrow \cos(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \Rightarrow 1 - \sin(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \\ &\Rightarrow 1 - x^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \Rightarrow \sqrt{1 - x^2} \leq \frac{\sin(x)}{x} \leq 1. \end{aligned}$$

Solution:

As $\lim_{n \rightarrow \infty} t_n = 0$, then there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, then $|t_n| \leq \frac{\pi}{2}$. Moreover, as $\sin(x)$ is an odd function, then

$$\frac{\sin(t_n)}{t_n} = \frac{\sin(|t_n|)}{|t_n|}.$$

Then, by the hint, $\forall n \geq N$,

$$\sqrt{1 - t_n^2} \leq \frac{\sin(|t_n|)}{|t_n|} = \frac{\sin(t_n)}{t_n} \leq 1.$$

As $\lim_{n \rightarrow \infty} t_n = 0$, then

$$\lim_{n \rightarrow \infty} \sqrt{1 - t_n^2} = 1.$$

[Prove this!! Hint: for $0 < x < 1$, $1 > \sqrt{1 - x^2} > 1 - x^2 > 1 - x$.]

Hence, the squeeze theorem for sequences implies the desired conclusion.

6. Show that if $\lim_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|} = 1$, then anything can happen for $\sum_{n=0}^{\infty} x_n$. That is, it is possible to find sequences (x_n) such that:

- (a) (x_n) is unbounded;
- (b) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ diverges;
- (c) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ converges absolutely;
- (d) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ converges but not absolutely.

For each item above, provide an example.

Solution:

- (a) Take $x_n = \log(n)$.
- (b) Take $x_n = \frac{1}{n}$.
- (c) Take $x_n = \frac{1}{n^2}$.
- (d) Take $x_n = (-1)^n \frac{1}{n}$.

7. For each of the following, determine whether the series is convergent or divergent.

- (a) $\sum_{k=1}^{\infty} \frac{\pi^k}{k \cdot 2^k}$
- (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+4} - \sqrt{n+2}}{n}$
- (c) $\sum_{k=2}^{\infty} \frac{k^2 - 1}{(k-1)^3}$

Solution:

- (a) We observe that

$$\frac{\pi^k}{k \cdot 2^k} = \left(\frac{\pi}{2}\right)^k \cdot \frac{1}{k} \geq \frac{1}{k} \geq 0$$

for all $k \geq 1$. So the series diverges by the comparison criterion, because the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

- (b) We observe that

$$0 \leq \frac{\sqrt{n+4} - \sqrt{n+2}}{n} = \frac{n+4 - (n+2)}{n(\sqrt{n+4} + \sqrt{n+2})} = \frac{2}{n(\sqrt{n+4} + \sqrt{n+2})} \leq \frac{2}{n(\sqrt{n} + \sqrt{n})} = \frac{1}{n^{\frac{3}{2}}}.$$

Therefore, the series is convergent by the comparison criterion, because the series $\sum_{k=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges as $\frac{3}{2} > 1$.

- (c) We observe that

$$\frac{k^2 - 1}{(k-1)^3} = \frac{k+1}{(k-1)^2} \geq \frac{1}{k} \geq 0.$$

So the series diverges by the comparison criterion, because the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

8. Using the definition, state if the following functions are injective, surjective or bijective. If the function is bijective, find the inverse function.

(a) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^5$

(b) $f : [0, \infty) \rightarrow \mathbb{R}, f(x) = \sqrt{x}$

Solution:

(a) Injective: we must show that if for some $x_1 \in x_2$ in the domain of f , $f(x_1) = f(x_2)$ then $x_1 = x_2$. We have

$$f(x_1) = f(x_2) \Rightarrow x_1^5 = x_2^5 \Rightarrow \sqrt[5]{x_1^5} = \sqrt[5]{x_2^5} \Rightarrow x_1 = x_2.$$

So f is injective.

Surjective: we must show that for any $y \in \mathbb{R}$ there exist x in the domain of f such that $f(x) = y$. For any given y it is enough to take $x = \sqrt[5]{y}$, then $f(x) = y$. So f is surjective.

Since f is both injective and surjective then it is bijective. The inverse function is given by $f^{-1}(x) = \sqrt[5]{x}$.

(b) Injective: We have

$$f(x_1) = f(x_2) \Rightarrow \sqrt{x_1} = \sqrt{x_2} \Rightarrow (\sqrt{x_1})^2 = (\sqrt{x_2})^2 \Rightarrow |x_1| = |x_2| \Rightarrow x_1 = x_2.$$

The last step is true because we know that $x_1, x_2 \in [0, \infty)$. So f is injective.

Surjective: This function is not surjective since there is no $x \in [0, \infty)$ that is mapped to negative numbers. Since the function is not surjective then it is not invertible.

9. For the two functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ below, find $g \circ f$ and $f \circ g$.

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}, \quad g(x) = \begin{cases} 2x-3 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$$

Solution:

For $g \circ f$: Note that if $x \geq 0$ then $f(x) \geq 1$, if $-1 < x < 0$ then $0 < f(x) < 1$ and when $x \leq -1$ then $f(x) \geq 1$ so

$$g \circ f(x) = \begin{cases} 2(x+1)-3 & \text{if } x \geq 0 \\ 1-(x^2) & \text{if } -1 < x < 0 \\ 2(x^2)-3 & \text{if } x \leq -1 \end{cases}$$

and for $f \circ g$: Note that if $x \geq 3/2$ then $g(x) \geq 0$, if $1 \leq x < 3/2$ then $-1 \leq g(x) < 0$ and when $x < 1$ then $g(x) > 0$ so

$$f \circ g(x) = \begin{cases} (2x-3)+1 & \text{if } x \geq 3/2 \\ (2x-3)^2 & \text{if } 1 \leq x < 3/2 \\ (1-x)+1 & \text{if } x < 1 \end{cases}$$

10. State if the following are true or false.

- (a) The function $f = \sqrt{1 - \cos x}$ is even.
- (b) There is no function which is both even and odd.
- (c) Let f be an odd function. If f is bijective, then f^{-1} is also odd.

Solution:

- (a) True. We have

$$f(-x) = \sqrt{1 - \cos(-x)} = \sqrt{1 - \cos(x)} = f(x)$$

- (b) False. If a function is both even and odd then we have:

$$f(-x) = f(x) = -f(x) \implies f(x) = -f(x) \implies f(x) = 0$$

So $f(x) = 0$ is a function that is both even and odd.

- (c) True. Note that $f \circ f^{-1} = id$, which means that $f \circ f^{-1}(y) = id(y) = y$. We have

$$-y = -y \implies id(-y) = -id(y) \implies f \circ f^{-1}(-y) = -f \circ f^{-1}(y)$$

But since f is an odd function we have that $f(-x) = -f(x)$. So we can write

$$\implies f(f^{-1}(-y)) = f(-f^{-1}(y))$$

Now since f is injective we know that if $f(x_1) = f(x_2)$ then $x_1 = x_2$, meaning

$$\implies f^{-1}(-y) = -f^{-1}(y)$$

This shows that f^{-1} is an odd function.

11. Given functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$, Determine the monotonicity (increasing or decreasing) of the composition $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$ in the following cases:

- (a) if f and g are both increasing.
- (b) if f and g are both decreasing.
- (c) if f is increasing and g is decreasing. What can we say about $f \circ g$?

Solution:

- (a) If f and g are both increasing, we have

$$x_1 \leq x_2 \implies f(x_1) \leq f(x_2) \implies g(f(x_1)) \leq g(f(x_2))$$

In the first step we used monotonicity of f and in the second step we used monotonicity of g . So $g \circ f$ is an increasing function.

- (b) If f and g are both decreasing, we have

$$x_1 \leq x_2 \implies f(x_1) \geq f(x_2) \implies g(f(x_1)) \leq g(f(x_2))$$

In the first step we used monotonicity of f and in the second step we used monotonicity of g . So $g \circ f$ is an increasing function.

(c) If f is increasing and g is decreasing, we have

$$x_1 \leq x_2 \implies f(x_1) \leq f(x_2) \implies g(f(x_1)) \geq g(f(x_2))$$

so $g \circ f$ is decreasing.

For $f \circ g$ we have

$$x_1 \leq x_2 \implies g(x_1) \geq g(x_2) \implies f(g(x_1)) \geq f(g(x_2))$$

This shows that $f \circ g$ is again decreasing. Hence, the composition of functions with opposite monotonicity is always decreasing.

12. Using the definition, state if the following functions are injective, surjective or bijective. If the function is bijective, find the inverse function.

(a) $f : \mathbb{R} \rightarrow [-1, 1]$, $f(x) = \sin x$

(b) $f : [0, \pi] \rightarrow [-1, 1]$, $f(x) = \cos x$

Solution:

(a) Injective: This function is not injective, because we can find $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. For example $\sin(0) = \sin(\pi) = 0$.

Surjective: For any given y it is enough to take $x = \arcsin(y)$ then $f(x) = y$. So f is surjective.

(b) Injective: We have

$$f(x_1) = f(x_2) \Rightarrow \cos(x_1) = \cos(x_2) \Rightarrow \arccos(\cos(x_1)) = \arccos(\cos(x_2)) \Rightarrow x_1 = x_2.$$

So f is injective.

Surjective: For any given $y \in [-1, 1]$ it is enough to take $x = \arccos(y)$, then $f(x) = y$. So f is surjective.

Since f is both injective and surjective then it is bijective. The inverse function is given by $f^{-1}(x) = \arccos(x)$.

13. For the two functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ below, find $g \circ f$ and $f \circ g$.

$$f(x) = \begin{cases} |2x - 1| & \text{if } x \geq -1 \\ -x(x + 2) & \text{if } x < -1 \end{cases}, \quad g(x) = \begin{cases} -\sqrt{x - 4} & \text{if } x \geq 4 \\ 1 - x/2 & \text{if } x < 4 \end{cases}$$

Solution:

For $f \circ g$:

For $x \geq 4$ we have

$$-\sqrt{x - 4} \geq -1 \Leftrightarrow x - 4 \leq 1 \Leftrightarrow 4 \leq x \leq 5$$

and for all $x < 4$ we have $1 - \frac{1}{2}x \geq -1$. So $g(x) \geq -1 \Leftrightarrow x \leq 5$ and it follows

$$\begin{aligned}(f \circ g)(x) &= \begin{cases} |2g(x) - 1|, & x \leq 5 \\ -g(x)(g(x) + 2), & x > 5 \end{cases} \\ &= \begin{cases} |1 - x|, & x < 4 \\ |-2\sqrt{x-4} - 1|, & 4 \leq x \leq 5 \\ 2\sqrt{x-4} - x + 4, & x > 5 \end{cases} = \begin{cases} |1 - x|, & x < 4 \\ 2\sqrt{x-4} + 1, & 4 \leq x \leq 5 \\ 2\sqrt{x-4} - x + 4, & x > 5 \end{cases}\end{aligned}$$

For $g \circ f$:

For $x \geq -1$, we have

$$|2x - 1| \geq 4 \Leftrightarrow 2x - 1 \geq 4 \text{ or } 2x - 1 \leq -4 \Leftrightarrow x \geq \frac{5}{2} \quad (\text{since } x \geq -1)$$

and for $x < -1$ we have $-x(x+2) \geq 4 \Leftrightarrow x^2 + 2x + 4 \leq 0$, which is impossible since the polynomial has no real roots. So $f(x) \geq 4 \Leftrightarrow x \geq \frac{5}{2}$ and therefore

$$\begin{aligned}(g \circ f)(x) &= \begin{cases} -\sqrt{f(x) - 4}, & x \geq \frac{5}{2} \\ 1 - \frac{1}{2}f(x), & x < \frac{5}{2} \end{cases} \\ &= \begin{cases} -\sqrt{|2x - 1| - 4}, & x \geq \frac{5}{2} \\ 1 - \frac{1}{2}|2x - 1|, & -1 \leq x < \frac{5}{2} \\ \frac{1}{2}x^2 + x + 1, & x < -1 \end{cases} = \begin{cases} -\sqrt{2x - 5}, & x \geq \frac{5}{2} \\ 1 - \frac{1}{2}|2x - 1|, & -1 \leq x < \frac{5}{2} \\ \frac{1}{2}x^2 + x + 1, & x < -1 \end{cases}\end{aligned}$$

14. State if the following are true or false.

- (a) If f is an even function and g is an odd function, then $h = f \cdot g$ is an odd function.
- (b) If f is an even function and g is an odd function, then $h = f \circ g$ is an odd function.
- (c) A function is either even or odd or both.

Solution:

- (a) True. We have

$$h(-x) = f(-x)g(-x) = (f(x)) \cdot (-g(x)) = -f(x)g(x) = -h(x)$$

- (b) False. We have

$$h(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = h(x)$$

So h is an even function.

- (c) False. Take for example $f(x) = x + x^2$. Then $f(-x) = -x + x^2 \neq f(x) \neq -f(x)$. So this function is neither even nor odd.

15. Calculate the following limits.

- (a) $\lim_{x \rightarrow 0} \frac{x^3 + 4x}{2x}$
 (b) $\lim_{x \rightarrow 0} \frac{\sqrt{9+x}-3}{x}$
 (c) $\lim_{x \rightarrow 0} \frac{\cos(x)-1}{x^2}$

Solution:

(a)

$$\lim_{x \rightarrow 0} \frac{x^3 + 4x}{2x} = \lim_{x \rightarrow 0} \frac{x(x^2 + 4)}{2x} = \lim_{x \rightarrow 0} \frac{x^2 + 4}{2} = 2$$

(b)

$$\lim_{x \rightarrow 0} \frac{\sqrt{9+x}-3}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{9+x}-3}{x} \cdot \frac{\sqrt{9+x}+3}{\sqrt{9+x}+3} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{9+x}+3)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{6}$$

(c)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos(x)-1}{x^2} &= \lim_{x \rightarrow 0} \frac{-2\sin^2(x/2)}{x^2} = \lim_{x \rightarrow 0} (-2) \cdot \frac{\sin(x/2)}{2 \cdot x/2} \cdot \frac{\sin(x/2)}{2 \cdot x/2} \\ &= \lim_{x \rightarrow 0} \left(-\frac{1}{2}\right) \cdot \frac{\sin(x/2)}{x/2} \cdot \frac{\sin(x/2)}{x/2} = -\frac{1}{2} \end{aligned}$$

16. Calculate the following limits.

- (a) $\lim_{x \rightarrow 4} \frac{x^2 + 5x - 36}{x^2 - 16}$
 (b) $\lim_{x \rightarrow 1} \frac{x^n - 1}{x - 1}$ (*Hint: Try to factorize $x - 1$ from the numerator.*)

Solution:

(a)

$$\lim_{x \rightarrow 4} \frac{x^2 + 5x - 36}{x^2 - 16} = \frac{(x-4)(x+9)}{(x-4)(x+4)} = \lim_{x \rightarrow 4} \frac{x+9}{x+4} = \frac{13}{8}$$

(b) We use the formula

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

to rewrite

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^n - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x^{n-1} + x^{n-2} + \cdots + x + 1) \\ &= n \end{aligned}$$

17. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{500}}{(1.0001)^n}$$

- (a) converges absolutely.
- (b) converges, but not absolutely.
- (c) approaches $+\infty$.
- (d) approaches $-\infty$.

Solution:

(a) is correct. To check absolute convergence we consider the sequence

$$a_n = \left| \frac{(-1)^n n^{500}}{(1.0001)^n} \right| = \frac{n^{500}}{(1.0001)^n}$$

Now we use the ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^{500}}{(1.0001)^{n+1}}}{\frac{n^{500}}{(1.0001)^n}} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{500} \frac{1}{1.0001} = \frac{1}{1.0001} \leq 1$$

This means that the series is absolutely convergent by Alembert's criterion.

18. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sqrt{n}+1} - \frac{n+1}{\sqrt{n+1}+1} \right)$$

- (a) diverges.
- (b) converges to $\frac{1}{2} - \frac{2}{\sqrt{2}+1}$.
- (c) converges to $\frac{1}{2}$.
- (d) converges to 0.

Solution:

We observe that for every $m > 0$, the sequence

$$S_m := \sum_{n=1}^m \left(\frac{n}{\sqrt{n}+1} - \frac{n+1}{\sqrt{n+1}+1} \right) = \frac{1}{2} - \frac{m+1}{\sqrt{m+1}+1}$$

approaches $-\infty$. Hence, the series diverges.

19. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n} \right)^n$$

is

- (a) divergent.
- (b) converges to e .
- (c) converges to e^{-1} .
- (d) converges to 1.

Solution:

(a) is correct. Take $a_n = (1 - \frac{1}{n})^n$. We know that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} \neq 0$$

If the sequence does not converge to zero then the series is divergent.

20. (Multiple choice) The limit

$$\lim_{n \rightarrow \infty} \frac{\sqrt[3]{1 - \frac{1}{n}} - 1}{\sqrt[4]{1 - \frac{1}{n}} - 1}$$

is

- (a) $\frac{3}{4}$
- (b) $\frac{4}{3}$
- (c) ∞
- (d) 0

Solution:

(b) is correct. We use the identity

$$(a^N - 1) = (a - 1)(a^{N-1} + a^{N-2} + \cdots + 1).$$

For $a = \sqrt[3]{1 - \frac{1}{n}}$ and $N = 3$ we have

$$\left(1 - \frac{1}{n}\right) - 1 = \left(\sqrt[3]{1 - \frac{1}{n}} - 1\right) \left(\sqrt[3]{\left(1 - \frac{1}{n}\right)^2} + \sqrt[3]{1 - \frac{1}{n}} + 1\right).$$

For $a = \sqrt[4]{1 - \frac{1}{n}}$ and $N = 4$ we have

$$\left(1 - \frac{1}{n}\right) - 1 = \left(\sqrt[4]{1 - \frac{1}{n}} - 1\right) \left(\sqrt[4]{\left(1 - \frac{1}{n}\right)^3} + \sqrt[4]{\left(1 - \frac{1}{n}\right)^2} + \sqrt[4]{1 - \frac{1}{n}} + 1\right)$$

so

$$\lim_{n \rightarrow \infty} \frac{\sqrt[3]{1 - \frac{1}{n}} - 1}{\sqrt[4]{1 - \frac{1}{n}} - 1} = \lim_{n \rightarrow \infty} \frac{(1 - \frac{1}{n}) - 1}{(1 - \frac{1}{n}) - 1} \cdot \frac{\sqrt[4]{(1 - \frac{1}{n})^3} + \sqrt[4]{(1 - \frac{1}{n})^2} + \sqrt[4]{1 - \frac{1}{n}} + 1}{\sqrt[3]{(1 - \frac{1}{n})^2} + \sqrt[3]{1 - \frac{1}{n}} + 1} = \frac{4}{3}$$

21. (Multiple choice) The limit

$$\lim_{n \rightarrow \infty} \left(\frac{n+2}{n}\right)^n \frac{n+2}{n+1}$$

is

- (a) e^2

- (b) e
- (c) ∞
- (d) 0

Solution:

(a) is correct. We have

$$\left(\frac{n+2}{n}\right)^n \frac{n+2}{n+1} = \left(\frac{n+1}{n}\right)^n \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n+1}\right)^{n+1}$$

and

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n+1}\right)^{n+1} = e^2$$

22. (Multiple choice) The limit

$$\lim_{n \rightarrow \infty} n^2 \cdot \sin\left(\frac{2n+3}{n^3}\right)$$

is

- (a) 0
- (b) $\frac{1}{2}$
- (c) 2
- (d) ∞

Solution:

(c) is correct. Note that

$$\sqrt{1-x^2} \leq \frac{\sin x}{x} \leq 1$$

so

$$\sqrt{1 - \left(\frac{2n+3}{n^3}\right)^2} \leq \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\left(\frac{2n+3}{n^3}\right)} \leq 1$$

Since $\lim_{n \rightarrow \infty} \left(\frac{2n+3}{n^3}\right) = 0$ then by the squeeze theorem we have that

$$\frac{\sin\left(\frac{2n+3}{n^3}\right)}{\left(\frac{2n+3}{n^3}\right)} = 1.$$

So for the original limit we can write

$$\lim_{n \rightarrow \infty} n^2 \cdot \sin\left(\frac{2n+3}{n^3}\right) = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\left(\frac{2n+3}{n^3}\right)} \cdot \left(\frac{2n+3}{n^3}\right) \cdot n^2 = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\left(\frac{2n+3}{n^3}\right)} \cdot \frac{2n^3 + 3n^2}{n^3} = 2$$