Analysis I (English)
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Analysis 1 - Exercise Set 6

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Compute, if they exists, the limits of the following sequences

   (a) $\sqrt[n]{\frac{3}{n}}$

   (b) $(-1)^n \left(\frac{n^2+1}{n-1}\right)$

   (c) $\frac{1}{n^2}\left(\sqrt{1+n+\pi n^2+\frac{\sin(n)}{n}}-1\right)$

   (d) $\sqrt[n]{n\log(n)}$ (*Hint*: $1<\log(n)<n$ for $n>3$)

   (e) $n^2\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1\right)$

   (f) $\left(\frac{n-1}{n}\right)^{n^2}$

   (g) $\sqrt[n]{\frac{2n}{3n^2-1}}$

   (h) $\frac{4n^2-2\pi}{-n^3+\sqrt{7}n}$

   (i) $\frac{(n+1)!}{n!-(n+1)!}$

   (j) $\frac{\sqrt{\frac{cos(n)}{n^2}+1}-1}{\sqrt{e-\frac{1}{n}}-\sqrt{e}}$

Solution:

(a)

$$\lim_{n\to\infty}\sqrt[n]{\frac{3}{n}}=\frac{\lim_{n\to\infty}\sqrt[n]{3}}{\lim_{n\to\infty}\sqrt[n]{n}}=\frac{1}{1}=1$$

(b) Let $a_n=(-1)^n\left(\frac{n^2+1}{n-1}\right)$. Remark that the sequence

$$x_n=|a_n|=\frac{n^2+1}{n-1}=\frac{1+n^{-2}}{n^{-1}-n^{-2}}$$

approaches $+\infty$. Then (a_n) is divergent. The subsequence $a_{2n}=x_{2n}$ converges to $+\infty$; on the other hand, the subsequence $a_{2n+1}=-x_{2n+1}$ converges to $-\infty$. We conclude that a_n does not admit a limit.

(c) We have

$$\frac{1}{n^2}\left(\sqrt{1+n+\pi n^2+\frac{\sin(n)}{n}}-1\right)=\sqrt{\frac{1}{n^4}+\frac{1}{n^3}+\frac{\pi}{n^2}+\frac{\sin(n)}{n^5}}-\frac{1}{n^2}$$

so the limit is zero. (Remark that $\lim_{n\to\infty}\frac{\sin(n)}{n^5}=\lim_{n\to\infty}\frac{\sin(n)}{n}\frac{n}{n^5}$ is zero, because it is the limit of the product of a bounded sequences with a sequences that converges to 0.)

(d) Using the hint, we have that

$$\sqrt[n]{n}<\sqrt[n]{n\log(n)}<\sqrt[n]{n^2}.$$

Applying the squeeze theorem, we get that the limit is 1.

(e) The issue is that $\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1$ convergence to zero, but we do not know the speed of convergence, so we do not know how to compare it with n^2. To this end, we kill the square root as usual writing

$$n^2\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1\right)=\frac{n^2\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1\right)\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}+1\right)}{\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}+1\right)}=$$

$$=\frac{n^2\left(\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}\right)}{\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}+1\right)}=\frac{n+\pi+\frac{\sin(n)}{n^3}}{\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}+1\right)}.$$

We conclude that the limit is $+\infty$, because the denominator is a convergent sequence with limit $\neq 0$, the numerator is the sum of a sequence (n) that approaches $+\infty$ with a sequence $(\pi+\frac{\sin(n)}{n^3})$ that is bounded below.

(f) We have

$$\left(\frac{n-1}{n}\right)^{n^2}=\left(\left(1-\frac{1}{n}\right)^n\right)^n$$

We know that $\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)^n\right)=e^{-1}$. Hence we expect that $\left(\frac{n-1}{n}\right)^{n^2}$ converges to 0. Let's prove it.

Let $\varepsilon>0$ such that $0<\varepsilon<\min\{e^{-1},1-e^{-1}\}$. Let $a=e^{-1}-\varepsilon$ and $b=e^{-1}+\varepsilon$. Then $0<a<e^{-1}<b<1$.

Since $\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)^n\right)=e^{-1}$, there exists an N such that for all $n>N$ we have

$$\left|\left(1-\frac{1}{n}\right)^n-e^{-1}\right|<\varepsilon,$$

which is equivalent to

$$a<\left(\frac{n-1}{n}\right)^n<b.$$

Then

$$a^n<\left(\frac{n-1}{n}\right)^{n^2}<b^n$$

and we conclude applying squeeze theorem.

(g) We have

$$\sqrt[n]{\frac{2n}{3n^2-1}} = \frac{1}{\sqrt[n]{n}}\sqrt[n]{\frac{2n^2}{3n^2-1}} = \frac{1}{\sqrt[n]{n}}\sqrt[n]{\frac{2}{3-\frac{1}{n^2}}}$$

both factors converge now to 1, so the limit is 1.

(h) The limit is 0.

(i) Dividing numerator and denominator by $n!$ we get that

$$\frac{(n+1)!}{n!-(n+1)!} = -\frac{n+1}{n}$$

so the limit is -1.

(j) Numerator and denominator both tend to zero, and we tackle the square root in the usual way, our sequence is indeed equal to

$$\frac{\left(\frac{\cos(n)}{n^2}+1-1\right)\left(\sqrt{e+\frac{1}{n}}+\sqrt{e}\right)}{\left(\sqrt{e}-\frac{1}{n}-\sqrt{e}\right)\left(\sqrt{\frac{\cos(n)}{n^2}+1}+1\right)}$$

Which in turn equals

$$\left(\frac{\cos(n)}{n}\right)\left(\frac{\sqrt{e+\frac{1}{n}}+\sqrt{e}}{\sqrt{\frac{\cos(n)}{n^2}+1}+1}\right)$$

the first factor goes to zero, the second converge to a real number (actually $\sqrt{e}$, but this does not matter), so the limit is zero.

2. Let $a, b \in \mathbb{R}_+$ and (x_n) be a sequence defined by the recurrence relation

$$x_{n+1} = ax_n^2 \quad x_0 = b.$$

(a) Show by induction that every element in the sequence (x_n) is given by

$$x_n = a^{2^n-1}b^{2^n}.$$

(b) Use part (a) to compute

$$\lim_{n\to+\infty} x_n.$$

Solution:

(a) For $n = 0$ we have

$$x_0 = a^{2^0-1}b^{2^0} = b$$

which is true. Assuming that $x_n = a^{2^n-1}b^{2^n}$ for some n, we have

$$\begin{aligned} x_{n+1} =& ax_n^2 \\ =& a \cdot \left(a^{2^n-1}b^{2^n}\right)^2 \\ =& a \cdot a^{2\cdot 2^n-2}b^{2\cdot 2^n} \\ =& a^{2^{n+1}-1}b^{2^{n+1}}. \end{aligned}$$

(b) To calculate the limit, we first rewrite x_n as

$$x_n = a^{2^n-1}b^{2^n} = \frac{(ab)^{2^n}}{a}$$

Now we have 3 different possibilities. If $ab = 1$ then

$$\lim_{n\to+\infty} x_n = \frac{(ab)^{2^n}}{a} = \frac{1}{a}$$

if $ab < 1$ then

$$\lim_{n\to+\infty} x_n = \frac{(ab)^{2^n}}{a} = 0$$

if $ab > 1$ then

$$\lim_{n\to+\infty} x_n = \frac{(ab)^{2^n}}{a} = +\infty$$

3. Show that the following recursive sequence is convergent and calculate the limit

$$a_n = \frac{7}{3} - \frac{1}{1+a_{n-1}}, \quad a_1 = 1.$$

Solution:

If the limit $\lim_{n\to\infty} a_n = a$ exists, it should satisfy the equation

$$a = \frac{7}{3} - \frac{1}{1+a} \quad \Leftrightarrow \quad 1 = \left(\frac{7}{3} - a\right)(1+a) \quad \Leftrightarrow \quad 0 = \frac{4}{3} + \frac{4}{3}a - a^2 \quad \Leftrightarrow$$
$$3a^2 - 4a - 4 = (3a+2)(a-2) = 0 \quad \Leftrightarrow \quad a = 2 \text{ or } a = -\frac{2}{3}.$$

We show by induction that $a_n \geq 0$ for all $n \in \mathbb{N}^*$. We have $a_1 = 1 \geq 0$. If $a_{n-1} \geq 0$, then

$$a_n = \frac{7}{3} - \frac{1}{1+a_{n-1}} \geq \frac{7}{3} - 1 = \frac{4}{3} \geq 0.$$

So the only possible limit for (a_n) is $a = 2$.

We also show (by induction) that 2 is an upper bound for (a_n). We have $a_1 = 1 \leq a$. If $0 \leq a_{n-1} \leq a$, we than have

$$a_n = \frac{7}{3} - \frac{1}{1+a_{n-1}} \leq \frac{7}{3} - \frac{1}{1+a} = 2 = a\,.$$

Showing that (a_n) is an increasing sequence: for $n \geq 2$ we have

$$a_n - a_{n-1} = \frac{7}{3} - \frac{1}{1+a_{n-1}} - a_{n-1} = \frac{4+4a_{n-1}-3a_{n-1}^2}{3(1+a_{n-1})} \geq 0$$

if and only if $4 + 4a_{n-1} - 3a_{n-1}^2 \geq 0$. The last inequality is true because $0 \leq a_{n-1} \leq 2$. Since $(a_n)_{n\geq 1}$ is increasing an bounded sequence then it is convergent with the limit $a = 2$.

4. This question is going to show that, whenever we have a sequence that is defined recursively,

we need to show that it converges, and that computing the candidates for the limit is not enough.

Consider the sequence defined as $a_1 = 10$, $a_{n+1} = a_n^2$ for $n \geq 1$.

(a) Show that, if the limit of (a_n) exists, then it is either 0 or 1.

(b) Show that (a_n) diverges to $+\infty$.

Solution:

(a) Assume that (a_n) converges, and let L denote its limit. Then, by what we saw in worksheet 5, also (a_{n+1}) converges to L. So, by the recursive relation, we have $L = L^2$. Then, the solutions to the equation $L^2 - L = 0$ are 0 and 1, showing the claim.

(b) We can use induction to show that $a_n = 10^{2^{n-1}}$. As it is well known that the geometric sequence (10^m) diverges to $+\infty$, then so does $(10^{2^{n-1}})$, since it can be regarded as a subsequence of (10^m).

5. Compute the limit of $a_n = \left(\frac{n+3}{n+1}\right)^n$ using subsequences. (*Hint: first, manipulate the definition of a_n so that it looks more to the sequence of a previous exercise, then use the subsequence with odd indices.*)

Solution:

Write

$$a_n = \left(\frac{n+3}{n+1}\right)^n = \left(1 + \frac{2}{n+1}\right)^n$$

so that

$$a_{2n+1} = \left(1 + \frac{1}{n+1}\right)^{2n+1} = \left(\left(1 + \frac{1}{n+1}\right)^{n+1}\right)^2 \left(1 + \frac{1}{n+1}\right)^{-1}$$

so a_{2n+1} converges to e^2. Now, we argue as in Exercise 17 in the previous sheet.

6. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

(a) If a sequence is not bounded above, it must be increasing.

(b) Any monotone sequence has a convergent subsequence.

(c) If (a_n) has no divergent subsequence, then (a_n) is convergent.

(d) If (a_n) is Cauchy convergent, then also $(|a_n|)$ is Cauchy convergent.

(e) If (a_n) is a Cauchy sequence, then the sequence $b_n = c \cdot a_n$, $c \neq 0$ is a Cauchy sequence.

(f) If (a_n) is Cauchy, there exists $\varepsilon > 0$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \in \mathbb{N}$.

(g) Any sequence has a convergent subsequence.

(h) If (a_n) and (b_n) are Cauchy sequences, then the sequence $c_n = a_n + b_n$ is a Cauchy sequence.

Solution:

(a) False. Take $a_n = 0$ if n is even and $a_n = n$ if n is odd.

(b) False. Take $a_n = n$.

(c) True. (a_n) is a subsequence of (a_n) itself. So by definition it should be convergent.

(d) True. It follows directly from the inequality:

$$\big||a_n| - |a_m|\big| \le |a_n - a_m|.$$

(e) True. A shortcut is to use that fact that, for a sequence of real numbers, being Cauchy is equivalent to be convergent, and the statement is of course true if we replace Cauchy with convergent. Let us now give the proof using just the definition of Cauchy sequence.

To show that (b_n) is Cauchy we must show that for any $\epsilon > 0$, there is N such that for all $i, j > N$, $|b_i - b_j| < \epsilon$. Let $\epsilon > 0$ be given. Since (a_n) is Cauchy, there is N such that for all $i, j > N$ we have

$$|a_i - a_j| < \frac{\epsilon}{|c|}$$

Now for all $i, j > N$ we have

$$|b_i - b_j| = |ca_i - ca_j| = |c| \cdot |a_i - a_j| \le |c| \cdot \frac{\epsilon}{|c|} = \epsilon.$$

So (b_n) is Cauchy.

(f) True. A sequence is Cauchy if and only if it is convergent. So (a_n) is a convergent sequence, let $a \in \mathbb{R}$ be its limit. Then we can find $C \in \mathbb{R}$ such that

$$|a_n - a| \le C$$

for all $n \in \mathbb{N}$. Using the triangle inequality we have

$$|a_m - a_n| \le |a_m - a| + |a - a_n| \le 2C$$

for all $m, n \in \mathbb{N}$. So it is enough to take $\varepsilon = 2C$.

(g) False. Take $a_n = n$.

(h) True. Since (a_n) and (b_n) are Cauchy sequences they must converge. By Algebra of Limits the sequence $c_n = a_n + b_n$ must converge, which is equivalent to c_n being a Cauchy sequence.

7. Show if the sequence

$$a_n = \frac{\sin(a_{n-1}) + 1}{2} \qquad a_1 = 0$$

satisfies the definition of Cauchy sequence. (*Hint: Use the trigonometric formulas from Exercise Sheet 1*)

Solution:
We have for all integers $n \geq 2$:

$$|a_{n+1} - a_n| = \frac{1}{2}\left|\sin(a_n) - \sin(a_{n-1})\right| = \frac{1}{2}\left|2\sin\left(\frac{a_n - a_{n-1}}{2}\right)\cos\left(\frac{a_n + a_{n-1}}{2}\right)\right|$$
$$\leq \left|\sin\left(\frac{a_n - a_{n-1}}{2}\right)\right| \leq \frac{|a_n - a_{n-1}|}{2} \ ,$$

In the last inequality we use the fact that $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$.
If we apply the above inequality $n-1$ times, we obtain

$$|a_{n+1} - a_n| \leq \frac{|a_2 - a_1|}{2^{n-1}} = \frac{\left|\frac{1}{2} - 0\right|}{2^{n-1}} = \frac{1}{2^n}\,.$$

For all couple of integers $n > m \geq 2$, by using the triangle inequality we obtain:

$$|a_n - a_m| \leq \sum_{k=m}^{n-1} |a_{k+1} - a_k|\,,$$

so

$$|a_n - a_m| \leq \sum_{k=m}^{n-1} \frac{1}{2^k} = \frac{1}{2^m}\sum_{k=0}^{n-m-1}\left(\frac{1}{2}\right)^k$$
$$= \frac{1}{2^m}\,\frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}} = \frac{1}{2^{m-1}}\underbrace{\left(1 - \left(\frac{1}{2}\right)^{n-m}\right)}_{\leq 1} \leq \frac{1}{2^{m-1}}\,.$$

Since the sequence $(\frac{1}{2^{m-1}})$ converges to 0, for every $\varepsilon > 0$ there exists $M_\varepsilon > 2$ such that for all $m \geq M_\varepsilon$ we have $\frac{1}{2^{m-1}} < \varepsilon$. Then $|a_n - a_m| \leq \varepsilon$ for all $n > m \geq M_\varepsilon$. We conclude that $|a_n - a_m| \leq \varepsilon$ for all $n, m \geq M_\varepsilon$. Therefore it follows that $(a_n)_{n\geq 1}$ is a Cauchy sequence.

8. Let (a_n) and (b_n) be two sequences. Show the following facts.

   (a) Assume that (a_n) and (b_n) are bounded. Prove that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

   (b) Provide an example of sequences (a_n) and (b_n) such that the inequality in part (a) is strict.

   (c) Assume that $\liminf a_n = 5$. Show that there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, $a_n \geq 4$.

   (d) Assume (b_n) is defined as follows:

$$b_n = \begin{cases} \frac{100}{n} \text{ if } 3|n \\ 2 - \frac{1}{n} \text{ if } 3|n-1 \\ \frac{1}{2} \text{ if } 3|n-2 \end{cases}$$

   Compute $\limsup b_n$, $\liminf b_n$, and exhibit a subsequence of (b_n) converging to $\limsup b_n$ and a subsequence converging to $\liminf b_n$.

Solution:

(a) We have that $\limsup a_n = \lim_{n\to\infty} \sup\{a_k | k \geq n\}$, $\limsup b_n = \lim_{n\to\infty} \sup\{b_k | k \geq n\}$, and $\limsup(a_n + b_n) = \lim_{n\to\infty} \sup\{a_k + b_k | k \geq n\}$ and all of these limits exist finite, as (a_n), (b_n), and therefore $(a_n + b_n)$ are bounded.

Fix $n \in \mathbb{N}$ and $h \geq n$. Then, $a_h + b_h \leq \sup\{a_k | k \geq n\} + b_h \leq \sup\{a_k | k \geq n\} + \sup\{b_k | k \geq n\}$, where we used the definition of sup and the fact that $h \geq n$. Thus, $\sup\{a_k | k \geq n\} + \sup\{b_k | k \geq n\}$ is an upper bound for $\{a_k + b_k | k \geq n\}$. As sup is the least upper bound of a set, it follows that $\sup\{a_k + b_k | k \geq n\} \leq \sup\{a_k | k \geq n\} + \sup\{b_k | k \geq n\}$.

(b) Consider $(a_n) = (-1)^n$ and $(b_n) = (-1)^{n+1}$. Then, $\limsup a_n = \limsup b_n = 1$, but $a_n + b_n = 0$ for all n. Thus, $\limsup(a_n + b_n) = 0 < 2 = \limsup a_n + \limsup b_n$.

(c) By definition of $\liminf$, we have $\lim_{n\to\infty} \inf\{a_k | k \geq n\} = 5$. Now, we apply the definition of convergence to the sequence $(\inf\{a_k | k \geq n\})$ with limit 5 and $\epsilon = 1$. Thus, there exist $N \in \mathbb{N}$ such that, for every $n \geq N$, we have $|\inf\{a_k | k \geq n\} - 5| \leq 1$. In particular, we have $\inf\{a_k | k \geq n\} \geq 5 - 1 = 4$. In particular, we have $\inf\{a_k | k \geq N\} \geq 4$. Then, by definition of inf, we have $a_k \geq 4$ for every $k \geq N$.

(d) Clearly, the sequence is bounded above by 100 and below by 0. So, lim sup and lim inf are finite. For $n \geq 50$, we have $\frac{100}{n} \leq 2$. Thus, for every $k \geq 50$, $b_k \leq 2$. So, by how the sequence is defined, for every $n \geq 50$, we have $\sup\{b_k | k \geq n\} \leq 2$. This shows that $\limsup b_n \leq 2$. On the other hand, since $\{k | k \geq n \text{ and } 3 | k - 1\} \subset \{k | k \geq n\}$, we have

$$\sup\{b_k | k \geq n\} \geq \sup\{b_k | k \geq n \text{ and } 3 | k - 1\} = \sup\{2 - \frac{1}{k} | k \geq n \text{ and } 3 | k - 1\} = 2.$$

So, we have $\limsup b_n = 2$. Notice that, for $n \geq 50$, the sequence $\sup\{b_k | k \geq n\}$ the sequence becomes constant with value 2. If we define $n_k = 3k + 1$, we have that (b_{n_k}) converges to 2, as $b_{n_k} = 2 - \frac{1}{3k+1}$.

Now, since $b_n \geq 0$ for every n, we have $\liminf b_n \geq 0$. We claim that the sequence $\inf\{b_k | k \geq n\}$ is constant with value 0. Indeed, we have

$$0 \leq \inf\{b_k | k \geq n\} \leq \sup\{b_k | k \geq n \text{ and } 3 | k\} = \inf\{\frac{100}{k} | k \geq n \text{ and } 3 | k\} = 0,$$

where the first inequality follows from the fact that (b_n) is non-negative, the second inequality from the fact that $\{k | k \geq n \text{ and } 3 | k\} \subset \{k | k \geq n\}$. This shows the claim and that $\liminf b_n = 0$. Then, if we define $m_k = 3_k$, the subsequence (b_{m_k}) converges to 0, as $b_{m_k} = \frac{100}{3k}$.

9. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

(a) If (x_n) is a sequence that converges to 0, then the series $\sum_{n=0}^{\infty} x_n$ converges.

(b) Let (x_n) and (y_n) be two sequences such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{\infty} x_n$ diverges, then the series $\sum_{n=0}^{\infty} y_n$ diverges.

(c) Let (x_n) and (y_n) be sequences such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{\infty} x_n$ diverges, then the series $\sum_{n=0}^{\infty} y_n$ diverges.

(d) Let (x_n) and (y_n) be sequences. If the series $\sum_{n=0}^{\infty} x_n$ converges and the sequence (y_n) converges, then the series $\sum_{n=0}^{\infty} x_n y_n$ converges.

Solution:

(a) False. Take $x_n = \frac{1}{n}$ for example.

(b) True. Let $S_n := \sum_{k=0}^{n} x_k$ and $T_n := \sum_{k=0}^{n} y_k$. Then $S_n \leq T_n$ for all $n \in \mathbb{N}$. Since $x_n, y_n \geq 0$ for all $n \in \mathbb{N}$, the sequences (S_n) and (T_n) are both monotone increasing. If $\sum_{n=0}^{\infty} x_n$ diverges, then (S_n) is not bounded above. Since $S_n \leq T_n$ for all $n \in \mathbb{N}$, then also (T_n) is unbounded above, in particular, it cannot converge.

(c) False. Take $x_n = -n$ and $y_n = \frac{1}{n^2}$ for example.

(d) False. Take for example $x_n = y_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\lim_{n\to\infty} y_n = 0$, the series $\sum_{n=0}^{\infty} x_n$ converges by the Leibniz criterion, and the series $\sum_{n=0}^{\infty} x_n y_n = \sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

10. For each of the following, determine whether the series is convergent or divergent.

(a) $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$

(b) $\sum_{n=0}^{\infty} \frac{2n^2+1}{3n^2+2}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

Solution:

(a) Define the sequence $a_n = \frac{1}{n^2+n+3}$ and the sequence $b_n = \frac{1}{n^2}$. We know that for all n we have that $a_n < b_n$. We also know that the series $\sum_{n=0}^{\infty} b_n$ converges. By the comparison criterion (see Exercise 9(b)), we conclude that the series $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$ converges.

(b) Define the sequence $a_n = \frac{2n^2+1}{3n^2+2}$, this sequence does not converge to zero. So the series is divergent.

(c) Define the sequence $a_n = \frac{1}{\sqrt{n+3}}$. This sequence converges to zero and is monotone. By the Leibniz criterion alternating series we can then directly conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ converge.

11. For each of the following, determine whether the series is convergent or divergent.

(a) $\sum_{n=0}^{\infty} \frac{\sin(2n^2)}{n^2+3}$

(b) $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+3}$

Solution:

(a) To figure this out define the new series by

$$\sum_{n=0}^{\infty} \left| \frac{\sin\left(2n^2\right)}{n^2+3} \right|$$

We know that absolute converge implies convergence. So if the above series converges, then the original series $\sum_{n=0}^{\infty} \frac{\sin(2n^2)}{n^2+3}$ must also converge. Define the sequence $a_n = \left|\frac{\sin(2n^2)}{n^2+3}\right|$ and the sequence $b_n = \frac{1}{n^2}$. We know that $a_n < b_n$ for all $n \geq 1$. We also know that the series $\sum_{n=0}^{\infty} b_n$ converges; by the comparison criterion (see Exercise 9(b)), we conclude that the series $\sum_{n=0}^{\infty} \frac{\sin(2n^2)}{n^2+3}$ converge.

(b) Applying the ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{10}{16} < 1$$

So this series is convergent by Alembert's criterion.

(c) Let $x_n := (-1)^n \frac{n}{n+3}$. The series diverges. Indeed, if it were convergent, then (x_n) would be convergent and have limit 0. Then also $\lim_{n\to\infty} |x_n|$ would be 0. This is in contradiction with

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} \frac{n}{n+3} = 1 \neq 0.$$

12. For each of the following, determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+3)}}$

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^5}}{n^3+1}$

(c) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

Solution:

(a) We observe that

$$0 \leq \frac{1}{\sqrt{n(n^2+3)}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{\frac{3}{2}}}.$$

Therefore, the series is convergent by the comparison criterion (see Exercise 9(b)), because the series $\sum_{k=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges as $\frac{3}{2} > 1$.

(b) We observe that

$$\frac{\sqrt{n^5}}{n^3+1} = \frac{1}{n^{\frac{1}{2}} + \frac{1}{n^{\frac{5}{2}}}}$$

is very close to $\frac{1}{n^{\frac{1}{2}}}$, so we expect that the series does not converge, because the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges. We also observe that $\frac{1}{n^{\frac{1}{2}}} \geq \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence we try to use the comparison criterion for the series we are studying. We observe that

$$\frac{1}{n^{\frac{1}{2}} + \frac{1}{n^{\frac{5}{2}}}} \geq \frac{1}{n} \tag{1}$$

holds if and only if

$$\frac{1}{n^{\frac{1}{2}}} + \frac{1}{n^{\frac{7}{2}}} \leq 1. \tag{2}$$

Since the sequence $\left(\frac{1}{n^{\frac{1}{2}}}+\frac{1}{n^{\frac{7}{2}}}\right)$ converges to 0, there exists $N>0$ such that (2) holds for all $n>N$. Then also (1) holds for all $n>N$. Hence, we conclude that the series $\sum_{n=1}^{\infty}\frac{\sqrt{n^5}}{n^3+1}$ diverges by the comparison criterion (see Exercise 9(b)), because the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

(c) We observe that

$$0\leq\frac{(k!)^2}{(2k)!}=\frac{k\cdot(k-1)\cdots 1}{2k\cdot(2k-1)\cdots(k+1)}=\prod_{j=0}^{k-1}\frac{k-j}{2k-j}=\prod_{j=0}^{k-1}\frac{1}{1+\frac{k}{k-j}}\leq\left(\frac{1}{2}\right)^k$$

because $\frac{k}{k-j}\geq 1$. Therefore, the series is convergent by the comparison criterion (see Exercise 9(b)), because the series $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ converges.

13. (Multiple choice) The series

$$\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$$

is

(a) divergent.

(b) converges to $2+\sqrt{2}$.

(c) converges to $2-\sqrt{2}$.

(d) cannot be determined.

Solution:

(b) is correct. We use the identity

$$a^N-1=(a-1)(a^{N-1}+a^{N-2}+\cdots+1)$$

to see that

$$S_N:=\sum_{n=0}^{N}\left(\frac{1}{\sqrt{2}}\right)^n=\frac{\left(\frac{1}{\sqrt{2}}\right)^{N+1}-1}{\frac{1}{\sqrt{2}}-1}$$

So

$$\lim_{N\to\infty}S_N=\frac{1}{1-\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{\sqrt{2}-1}=\sqrt{2}(\sqrt{2}+1)=2+\sqrt{2}.$$

14. (Multiple choice) The series

$$\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$$

(a) converges absolutely.

(b) converges, but not absolutely.

(c) diverges to $+\infty$.

(d) diverges to $-\infty$.

Solution:

(b) is correct. The point of this problem is really just to acknowledge that it is in fact an alternating series. To see this we need to acknowledge that,

$$\cos(n\pi) = (-1)^n$$

and so the series is really,

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Now define a sequence $b_n = \frac{1}{\sqrt{n}}$. The series is convergent, if $\lim_{n\to\infty} b_n = 0$ and b_n is monotone. We see that indeed:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

and also b_n is a decreasing sequence. But on the other hand

$$\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi)}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = +\infty$$

So the series is convergent, but not absolutely.

15. Terminate the proof that we started in class showing the convergence of $\sum_{i=1}^{\infty} \frac{(-1)^i}{i}$. This is what we have proven in class and that you can assume:

   (a) the subsequence (y_k) of (s_n),

   $$y_k := s_{2k+1} = \sum_{i=0}^{2k+1} \frac{(-1)^i}{i}$$

   is strictly increasing;

   (b) (y_k) is bounded; in particular (y_k) converges to a limit $y \in \mathbb{R}$.

(Hint: Show that (s_n) is a Cauchy sequence. Use the fact that since (y_k) converges, then it is Cauchy, and that $s_{2k} - \frac{1}{2k+1} = s_{2k+1}$.)

Solution:

Since (y_k) converges, in particular it is a Cauchy sequence. Hence for any $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that for all $k_1, k_2 \in \mathbb{N}$, $k_1, k_2 \geq k_\epsilon$,

$$|y_{k_1} - y_{k_2}| < \epsilon.$$

But this is equivalent to saying that for any $\epsilon > 0$ there exists $n'_\epsilon \in \mathbb{N}$ such that for all $n'_1, n'_2 \in \mathbb{N}$, n'_1, n'_2 **odd**, $n'_1, n'_2 \geq n'_\epsilon$,

$$|s_{n'_1} - s_{n'_2}| < \epsilon.$$

We have to find n_ϵ such that

$$|s_{n_1} - s_{n_2}| < \epsilon.$$

holds for any $n_1, n_2 \geq n_\epsilon$ and not just the odd ones, for any fixed value of ϵ.
If n_1 is even and n_2 is odd, then we can write

$$|s_{n_1} - s_{n_2}| = |s_{n_1-1} + \frac{1}{n_1} - s_{n_2}| \leq \frac{1}{n_1} + |s_{n_1-1} - s_{n_2}|,$$

where the inequality follows from the triangle inequality. If n_1, n_2 are even, then we can write

$$|s_{n_1} - s_{n_2}| = |s_{n_1-1} + \frac{1}{n_1} - s_{n_2-1} + \frac{1}{n_2}| \leq \frac{1}{n_1} + \frac{1}{n_2} + |s_{n_1-1} - s_{n_2-1}|,$$

where the inequality follows from the triangle inequality.
Fix $\epsilon > 0$. Let $n''_\epsilon \in \mathbb{N}$ be such that $\frac{1}{n''_\epsilon} < \frac{\epsilon}{4}$. Let $n'''_\epsilon := n'_{\frac{\epsilon}{2}}$ be the index in the definition of Cauchy sequence for (y_k) with $\frac{\epsilon}{2}$. Define $n_\epsilon = \max n''_\epsilon, n'''_\epsilon$. Then for natural numbers $n, m, r, t \geq n_\epsilon$, r, t odd, we have that

$$\frac{1}{n}, \frac{1}{m} \leq \frac{\epsilon}{4},$$
$$|s_r - s_t| < \frac{\epsilon}{2}.$$

If n_1 is even and n_2 is odd, $n_1, n_2 \geq n_\epsilon$ then we can write

$$|s_{n_1} - s_{n_2}| = |s_{n_1-1} + \frac{1}{n_1} - s_{n_2}| \leq \frac{1}{n_1} + |s_{n_1-1} - s_{n_2}| < \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon.$$

If n_1, n_2 are even, then we can write

$$|s_{n_1} - s_{n_2}| = |s_{n_1-1} + \frac{1}{n_1} - s_{n_2-1} + \frac{1}{n_2}| \leq \frac{1}{n_1} + \frac{1}{n_2} + |s_{n_1-1} - s_{n_2-1}| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$