Analysis I (English)
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Analysis 1 - Exercise Set 5

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. If a sequence (x_n) converges, then its limit is unique.

Solution: Let (x_n) be a convergent sequence, and assume that both $x, y \in \mathbb{R}$ are limits of this sequence. We need to conclude that $x = y$.

By definition of convergence, for every $0 < \varepsilon \in \mathbb{R}$ there are $n^x_\varepsilon, n^y_\varepsilon \in \mathbb{N}$ such that for all $n \geq n^x_\varepsilon$ we have:

$$|x - x_n| \leq \varepsilon$$

and for all $n \geq n^y_\varepsilon$ we have

$$|y - x_n| \leq \varepsilon.$$

So, if we set $n_\varepsilon := \max\{n^x_\varepsilon, n^y_\varepsilon\}$, then both of the above inequalities hold for all integers $n \geq n_\varepsilon$. In particular, for such n, we have

$$|y - x| = \underbrace{|y - x_n + x_n - x| \leq |y - x_n| + |x_n - x|}_{\text{triangle inequality}} \leq \varepsilon + \varepsilon = 2\varepsilon$$

Since, this holds for all $0 < \varepsilon \in \mathbb{R}$, we obtain that $y = x$.

2. Assume that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. Prove the following fact: for any $l \in \mathbb{N}$, $\lim_{n\to\infty} x_{n+l}$ exists and $\lim_{n\to\infty} x_{n+l} = x$.

Solution: Let us fix $l \in \mathbb{N}$. Let us define the sequence (y_n), $y_n := x_{n+l}$. Then, we need to prove that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$. If $l = 0$, then there is nothing to prove, since $x_n = y_n$, $\forall n \in \mathbb{N}$. Hence, we can assume that $l > 0$. By definition of limit, for any $\varepsilon > 0$ there exists $n'_\varepsilon > 0$ such that

$$\forall n \geq n'_\varepsilon, \text{ then } |x_n - x| < \varepsilon.$$

As $x_n = y_{n-l}$, then

$$\forall n \geq n'_\varepsilon, \text{ then } |y_{n-l} - x| < \varepsilon.$$

Hence, we can rewrite the above by saying that $\forall n \geq n'_\varepsilon - l$, then $|y_n - x| < \varepsilon$. Hence, taking $n_\varepsilon := n'_\varepsilon - l$, we see that x satisfies the definition of limit for y_n.

3. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

(a) If
$$\lim_{n\to\infty} a_n = 0,$$
then
$$\lim_{n\to\infty} (a_n \sin(n)) = 0.$$

(b) If (a_n) is bounded, then
$$\lim_{n\to\infty} (a_n e^{-n}) = 0.$$

(c) If
$$\lim_{n\to\infty} a_n = 0,$$
then the sequence $b_n := a_n e^n$ is unbounded.

Solution:

(a) True. Note $-a_n \le a_n \sin(n) \le a_n$ for all n, as the sin function is bounded in $[-1, 1]$. The result follows from applying the Squeeze Theorem.

(b) True. By the boundedness of the sequence $\exists C > 0$ s.t. $-Ce^{-n} \le a_n e^{-n} \le Ce^{-n}$. The result follows from applying the Squeeze Theorem, as $\lim\limits_{n\to\infty} e^{-n} = 0$.

(c) False. Take $a_n = e^{-n}$. Then $b_n = 1,\ \forall n \in \mathbb{N}$.

4. Compute the following limits:

(a) $\lim\limits_{n\to\infty} \frac{2^n - 3^n}{3^n + 1}$

(b) $\lim\limits_{n\to\infty} n^3 \left(1 - \cos(\frac{1}{n})\right) \sin(\frac{1}{n})$

(*Hint: Use the fact that* $\lim\limits_{m\to\infty} \frac{\sin(\frac{1}{m})}{\frac{1}{m}} = 1$ *and* $\lim\limits_{m\to\infty} \cos\left(\frac{1}{m}\right) = 1$.)

(c) $\lim\limits_{n\to\infty} \frac{\sin^2(n)}{2^n}$

(d) $\lim\limits_{n\to\infty} n(\sqrt{n^4 + 6n + 3} - n^2)$

Solution:

(a)
$$\lim_{n\to\infty} \frac{2^n - 3^n}{3^n + 1} = \lim_{n\to\infty} \frac{3^n((\frac{2}{3})^n - 1)}{3^n(1 + \frac{1}{3^n})} = \lim_{n\to\infty} \frac{(\frac{2}{3})^n - 1}{1 + \left((\frac{1}{3}\right)^n}.$$

The geometric sequences $\left(\frac{2}{3}\right)^n$ and $\left(\frac{1}{3}\right)^n$ converge to 0, as their ratio is strictly between 0 and 1. Thus, the numerator converges to -1, and the denominator to 1; since the limit of the denominator is not zero, the ratio converges to the ration of the limits which is -1.

(b) We use the equalities
$$\sin^2\left(\frac{1}{n}\right) = 1 - \cos^2\left(\frac{1}{n}\right) = \left(1 - \cos\left(\frac{1}{n}\right)\right)\left(1 + \cos\left(\frac{1}{n}\right)\right)$$

to get

$$\lim_{n\to\infty} n^3\left(1-\cos\left(\frac{1}{n}\right)\right)\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty}\left(1+\cos\left(\frac{1}{n}\right)\right)^{-1}\frac{\sin^3\left(\frac{1}{n}\right)}{n^{-3}} = \frac{1}{2},$$

where we use the two facts given in the hint to get that the limit of the first factor is $\frac{1}{2}$ and of the second factor is 1.

(c) We have

$$0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$$

and by Squeeze Theorem $\lim_{n\to\infty}\frac{\sin^2 n}{2^n} = 0$.

(d)

$$\begin{aligned}\lim_{n\to\infty} n(\sqrt{n^4+6n+3}-n^2) &= \lim_{n\to\infty} n(\sqrt{n^4+6n+3}-n^2)\frac{\sqrt{n^4+6n+3}+n^2}{\sqrt{n^4+6n+3}+n^2}\\ &= \lim_{n\to\infty}\frac{n(n^4+6n+3-n^4)}{\sqrt{n^4+6n+3}+n^2}\\ &= \lim_{n\to\infty}\frac{n(6n+3)}{n^2(\sqrt{1+6/n^3+3/n^4}+1)} = \frac{6}{2} = 3.\end{aligned}$$

5. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

(a) If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$$

then (a_n) converges.

(b) If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$$

then (a_n) diverges.

Solution:

(a) False, take $a_n = n$.

(b) False, take for example $a_n = 1/n$. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1$$

but $\lim\limits_{n\to\infty}\frac{1}{n} = 0$.

6. Determine if the sequence (a_n) is convergent or not in the following cases.

1. $a_n = \frac{n}{e^n}$.

2. $a_n = \frac{10^n}{n!}$
3. $a_n = \frac{n^n}{e^n}$
4. $a_n = \frac{n!}{n^n e^{\frac{n}{2}}}$

Solution:

1. We compute
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{e^{n+1}}\frac{e^n}{n} = \frac{n+1}{n}\frac{1}{e}.$$
Hence, $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{e} < 1$ and the sequence (a_n) is convergent by the quotient criterion.

2. We compute
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{(n+1)!}\frac{n!}{10^n} = \frac{10}{n+1}.$$
Hence, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ and the sequence (a_n) is convergent by the quotient criterion.

3. We compute
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{e^{n+1}}\frac{e^n}{n^n} = \frac{1}{e}\frac{(n+1)^{n+1}}{n^n} = \frac{1}{e}\left(\frac{n+1}{n}\right)^n (n+1).$$
Hence, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} n+1 = +\infty$ and the sequence (a_n) is unbounded by the quotient criterion.

4. We compute
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{(n+1)!}{(n+1)^{n+1}e^{\frac{n+1}{2}}}}{\frac{n!}{n^n e^{\frac{n}{2}}}} = \frac{(n+1)!}{(n+1)^{n+1}e^{\frac{n+1}{2}}}\cdot\frac{n^n e^{\frac{n}{2}}}{n!}\\ &= \frac{(n+1)!}{n!}\cdot\frac{n^n}{(n+1)^{n+1}}\cdot\frac{e^{\frac{n}{2}}}{e^{\frac{n+1}{2}}} = \frac{n+1}{n+1}\cdot\frac{n^n}{(n+1)^n}\cdot\frac{1}{e^{\frac{1}{2}}} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{e^{1/2}}.\end{aligned}$$
Then, the sequence $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ is convergent because $\left(\left(1+\frac{1}{n}\right)^n\right)$ is a convergent sequence with limit $\neq 0$. Then
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{e^{1/2}} = \frac{1}{e^{3/2}} < 1$$
and the sequence (a_n) is convergent by the quotient criterion.

5.

7. Determine if the following sequences converge or not. If the sequence is convergent, determine its limit.

   (a) $a_n = \frac{3n^2-1}{10n+5n^2}$

(b) $a_n = \frac{3^{2n}}{n}$

(c) $a_n = \frac{(-1)^n n^2}{2^n}$

Solution:

(a) Since the numerator and the denominator consist of polynomials of degree 2, then the sequence is convergent. We have

$$\lim_{n\to\infty} \frac{3n^2-1}{10n+5n^2} = \lim_{n\to\infty} \frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)} = \frac{3}{5}.$$

(b) We use the quotient criterion:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{3^{2(n+1)}}{n+1}}{\frac{3^{2n}}{n}} = 3^2 \lim_{n\to\infty} \frac{n}{n+1} = 3^2 > 1.$$

Since $\rho > 1$ the sequence is divergent.

(c) We use the quotient criterion:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{1}{2} \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = \frac{1}{2} < 1.$$

Since $\rho < 1$ the sequence (a_n) is convergent and the limit is 0.

8. Compute the following limits:

(a) $\lim_{n\to\infty} \frac{n^3}{7^n} \cos\left(n^2\right)$

(b) $\lim_{n\to\infty} \dfrac{\sin(n+1)-\sin(n-1)}{\cos(n+1)+\cos(n-1)}$

(*Hint: Use trigonometric formulas from Exercise Sheet 1*)

(c) $\lim_{n\to\infty} \dfrac{\sin\left(\sqrt{n^3+n^2+1}\right)}{n^3+n^2+1}$

Solution:

(a) First consider the sequence $b_n = \frac{n^3}{7^n}$. By induction we can show that $7^n \geq n^4$ for all $n \geq 6$ (so, the base case will be $n = 6$). The base case $n = 6$ is satisfied by direct inspection. Now suppose that for some $N \geq 6$ we know that $7^N \geq N^4$. We have

$$(N+1)^4 < 2N^4 < 2 \cdot 7^N < 7^{N+1},$$

as the first inequality is true for all $N \geq 6$; induction proof is complete. Then,

$$-\frac{n^3}{7^n} \leq \frac{n^3}{7^n}\cos(n^2) \leq \frac{n^3}{7^n} \implies -\frac{n^3}{n^4} \leq \frac{n^3}{7^n}\cos(n^2) \leq \frac{n^3}{n^4} \implies -\frac{1}{n} \leq \frac{n^3}{7^n}\cos(n^2) \leq \frac{1}{n}.$$

By the Squeeze Theorem we see that $\lim_{n\to\infty} a_n = 0$.

(b) We use the trigonometric formulas in Exercise Sheet 1.

$$\lim_{n\to\infty} \frac{\sin(n+1)-\sin(n-1)}{\cos(n+1)+\cos(n-1)} = \lim_{n\to\infty} \frac{2\cos(n)\sin(1)}{2\cos(n)\cos(1)} = \tan(1) \ .$$

(c) We have

$$\lim_{n\to\infty} \frac{\sin\left(\sqrt{n^3+n^2+1}\right)}{n^3+n^2+1} = 0 \ ,$$

because $\left|\sin\left(\sqrt{n^3+n^2+1}\right)\right| \leq 1$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{1}{n^3+n^2+1} = 0 \ .$$

9. Give an example of a sequence (x_n) such that the sequence $y_n = x_{n+1} - x_n$ converges to 0 but (x_n) itself is divergent.

Solution:

Take the sequence $x_n = \sqrt{n}$, $n \in \mathbb{N}$. x_n is clearly divergent but the sequence $y_n = \sqrt{n+1} - \sqrt{n}$ converges to zero:

$$\lim_{n\to+\infty} y_n = \lim_{n\to\infty} y_n \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = 0.$$

10. Prove that if $\lim\limits_{n\to\infty} x_n = +\infty$ and (y_n) bounded from below, then $\lim\limits_{n\to\infty}(x_n + y_n) = +\infty$. Show that this is true also if $+\infty$ is replaced with $-\infty$ and (y_n) is assumed to be bounded from above.

Solution:

Let C be a lower bound for the values of (y_n), that is, $C \leq y_n$, $\forall n \in \mathbb{N}$. Now, we need to show that, for every $M > 0$, there exists $N \in \mathbb{N}$ such that, if $n \geq N$, then $x_n + y_n \geq M$. Fix $M > 0$, and consider $M + |C| > 0$. As $\lim\limits_{n\to\infty} x_n = +\infty$, there exists N such that, if $n \geq N$, then $x_n \geq M + |C|$. Then, for every $n \geq N$, we have $x_n + y_n \geq M + |C| + C \geq M$. Thus, $x_n + y_n \to +\infty$ as $n \to \infty$ as required.

Now, we address the second part of the question. Let C be an upper bound for the values of (y_n), that is, $C \geq y_n \quad \forall n \in \mathbb{N}$. Then $x_n + y_n \leq x_n + C \to -\infty$ as $n \to \infty$ and thus $x_n + y_n \to -\infty$ as $n \to \infty$ as required.

11. Prove that if $x_n \neq 0$, for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} |\frac{x_n}{x_{n-1}}| = +\infty$ then (x_n) is unbounded and, thus it diverges. Construct examples of sequences (x_n) satisfying the conditions above and such that $\lim\limits_{n\to\infty} x_n = +\infty$ (resp. $\lim\limits_{n\to\infty} x_n = -\infty$).

Solution: By definition of divergence to $+\infty$, we have the following: for every $M > 0$, there exists $N \in \mathbb{N}$ such that, if $n \geq N$, then $\left|\frac{x_n}{x_{n-1}}\right| \geq C$. Now, let $C = 2$ and fix N accordingly. Then, for every $n \geq N$, by clearing denominators, we have $|x_n| \geq 2|x_{n-1}|$.

By induction, we show the following: for every $n \geq N$, $|x_n| \geq 2^{n-N}|x_N|$. We check the base case $n = N$: $|x_N| \geq |x_N| = 2^0|x_N| = 2^{N-N}|x_N|$. Now, fix $n \geq N$, and assume $|x_n| \geq 2^{n-N}|x_N|$. By what we showed in the previous paragraph, as $n+1 \geq N$, we have $|x_{n+1}| \geq 2|x_n|$. Applying the inductive hypothesis to $|x_n|$, we get $|x_{n+1}| \geq 2|x_n| \geq 2 \cdot 2^{n-N}|x_N| = 2^{n+1-N}|x_N|$. So, the inductive step is settled.

Since $|x_N| \neq 0$ and $\lim\limits_{m\to\infty} 2^m = +\infty$, we get that $\lim\limits_{n\to\infty} 2^{n-N}|x_N| = +\infty$. Since $|x_n| \geq 2^{n-N}|x_N|$ for every $n \geq N$, it follows that $\lim\limits_{n\to\infty} |x_n| = +\infty$. In particular, (x_n) is not bounded.

Examples are $x_n = e^{n^n}$ or $x_n = -e^{n^n}$ or $x_n = n!$ or $x_n = -n!$ among others.

12. Consider the recursive sequence $a_{n+1} = 7 - \frac{10}{a_n}$, with initial datum $a_1 = 4$. Compute the first three values. Then, show that it is bounded by 2 and 5, that it is increasing, and then compute the limit.

Solution:

The first values are 4, 4.5 and 4.778. We prove that it is bounded by 2 and 5 using induction. The claim is true for a_1. Now we have tho show that if a_n is bounded by 2 and 5, the same is true for a_{n+1}; this is done by explicit computation as follows:

$$a_{n+1} = 7 - \frac{10}{a_n} < 7 - \frac{10}{5} = 5, \quad a_{n+1} = 7 - \frac{10}{a_n} > 7 - \frac{10}{2} = 2.$$

To show that the sequence is increasing, we have to show that $a_n - a_{n+1}$ is non-positive for all n. This difference is equal to $a_n - 7 + \frac{10}{a_n}$, and it is negative for all a_n between 2 and 5 (solve the inequality $x - 7 + \frac{10}{x} < 0$). Since a_n is always between 2 and 5, we get the claim.

By monotone convergence we now know that the limit exists. Let us call it L. Taking the limit on both side of the equality $a_{n+1} = 7 - \frac{10}{a_n}$, we get that

$$L = 7 - \frac{10}{L}$$

So L is equal either to 2 or to 5. But now we recall that the sequence is increasing, and starts at 4, so $L = 5$.

13. Consider the recursive sequence $a_{n+1} = \sqrt{8a_n - 7}$, with initial datum $a_1 = 4$. Show that it is bounded by 1 and 7, that it is increasing, and then compute the limit.

Solution:

Same strategy as in Exercise 12. We prove by induction that the sequence is bounded by 1 and 7. We observe that $a_1 = 4$ is between 1 and 7. We assume that $1 \leq a_n \leq 7$ and we prove the same inequalities for a_{n+1} as follows:

$$a_{n+1} = \sqrt{8a_n - 7} \leq \sqrt{8 \cdot 7 - 7} = 7, \quad a_{n+1} = \sqrt{8a_n - 7} \geq \sqrt{8 \cdot 1 - 7} = 1.$$

The sequence is increasing if $a_{n+1} - a_n$ is non-negative. We observe that

$$a_{n+1} - a_n = \frac{a_{n+1}^2 - a_n^2}{a_{n+1} + a_n} = \frac{8a_n - 7 - a_n^2}{a_{n+1} + a_n}$$

is positive for a_n between 1 and 7. Hence, the sequence is increasing.

We know that the limit exists by monotone convergence. Let $L = \lim_{n\to\infty} a_n$. Then by taking the limit of both sides of the equation $a_{n+1}^2 = 8a_n - 7$ we get $L^2 = 8L - 7$. Hence, $L \in \{1, 7\}$. Since a_n is increasing and $a_1 = 4$, we conclude that $L = 7$.

14. Find the limit for $x_n = \frac{\sin(x_{n-1})}{2}, x_0 = 1$. [*Hint:* use the fact $|\sin(x)| \leq |x|$ for all x]

Solution:

Note $|x_n| = |\frac{\sin(x_{n-1})}{2}| \leq |\frac{x_{n-1}}{2}|$. Thus, by induction, we have that $|x_n| \leq \frac{|x_0|}{2^n} = \frac{1}{2^n}$ for all n. Since $\lim_{n\to\infty} \frac{1}{2^n} = 0$, by the squeeze theorem, we conclude $\lim_{n\to 0} x_n = 0$.

15. Let $(a_n), (b_n)$ be sequences. State if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

(a) If (a_n) is monotone, then $\lim_{n\to\infty} a_n$ exists or $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$.

(b) If (a_n) and (b_n) are monotone, then the sequence $c_n = a_n + b_n$ is monotone.

(c) If $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$, then (a_n) is a bounded sequence.

(d) An unbounded sequence can have a convergent subsequence.

(e) If (a_n) has no convergent subsequence, then (a_n) is unbounded.

Solution:

(a) True. Indeed, if (a_n) is monotone and bounded, then it converges and hence $\lim_{n\to\infty} a_n$ exists. If it is monotone increasing and unbounded, then it approaches $+\infty$ and so $\lim_{n\to\infty} a_n = +\infty$. If it is monotone decreasing and unbounded, then it approaches $-\infty$ and so $\lim_{n\to\infty} a_n = -\infty$.

(b) False. For example, take $a_n = 2n + (-1)^n$ and $b_n = -2n$. Then (a_n) is monotone increasing because

$$a_{n+1} - a_n = 2(n+1) + (-1)^{n+1} - 2n - (-1)^n = 2 + 2(-1)^{n+1} \geq 2 - 2 = 0,$$

and (b_n) is monotone decreasing. But $c_n = a_n + b_n = (-1)^n$ is not monotone.

(c) False. Take for example $a_n = \sqrt{n}$ for all $n \in \mathbb{N}$. Then

$$|a_{n+1} - a_n| = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

converges to 0 but (a_n) is not bounded.

(d) True. Take for example

$$a_n = \begin{cases} n^2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

then (a_{2k}) is a convergent subsequence, while a_n is divergent.

(e) True. This statement is the contrapositive of the Bolzano–Weierstrass theorem.

16. Compute the following limits:

(a) $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n$

(*Hint:* $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$, *and* $e = \lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1} = \lim_{n\to\infty} \frac{(n+2)^{n+1}}{(n+1)^{n+1}}$)

(b) $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n$

(*Hint:* $e^{-1} = \left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right)^{-1} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^{-n}\right) = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$)

(c) $\lim_{n\to\infty} \left(1-\frac{1}{n^2}\right)^n$

(*Hint:* $\left(1-\frac{1}{n^2}\right) = \left(1^2 - \left(\frac{1}{n}\right)^2\right)$)

Solution:

(a) Later in the course you will learn to do this with a change of variables; now you can do as follows

$$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty} \left(\frac{n+2}{n}\right)^n = \lim_{n\to\infty} \left(\frac{n+2}{n}\right)^n \left(\frac{n+2}{n+2}\right) \left(\frac{n+1}{n+1}\right)^{n+1}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \left(\frac{n+2}{n+1}\right)^{n+1} \left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1}\right)^{n+1} \left(\frac{n+1}{n+2}\right) = e^2$$

(b)

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \frac{n-1}{n} \frac{1}{\left(1+\frac{1}{n-1}\right)^{n-1}} = \frac{1}{e}$$

(c)

$$\lim_{n\to\infty} \left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)^n = 1$$

17. Let $a_n = \left(1+\frac{2}{n}\right)^n$. We are going to compute its limit using subsequences

(a) Compute $\lim_{n\to\infty} a_{2n}$.

(b) Show that $a_n \leq a_{n+1}$.

(c) Use subsequences, squeeze theorem and monotone convergence to show that the sequence (a_n) is convergent and its limit.

Solution:

(a) We have

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^2 = e^2.$$

(b) This is very similar to the proof that

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

You can find it in any analysis book, and in the lecture notes (Example 4.51, Introduction of e).

(c) We already know that the subsequence a_{2n} converges to e^2. Point (b) implies that $a_{2n} < a_{2n+1} < a_{2n+2}$. We can now use the squeeze theorem and point (a) to show that the sequence (a_{2n+1}) converges to e^2 as well. In particular

$$\sup\{a_{2n} : n \in \mathbb{N}\} = \sup\{a_{2n+1} : n \in \mathbb{N}\} = e^2$$

because both sequences (a_{2n}) and (a_{2n+1}) are monotone increasing and converge. Then (a_n) is bounded above and

$$\sup\{a_n : n \in \mathbb{N}\} = \max\{\sup\{a_{2n} : n \in \mathbb{N}\}, \sup\{a_{2n+1} : n \in \mathbb{N}\}\} = e^2.$$

Since the sequence (a_n) is monotone increasing and bounded above it converges by monotone convergence and the limit, which is the supremum, is e^2.