Analysis 1 - Exercise Set 4

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Prove that for all $n \in \mathbb{N}$, $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$.
   (b) Find all the solutions to the equation $x^n = 1$, for $n \in \mathbb{N}$.

Solution:

(a) We use the exponential representation of complex numbers:

$$(\cos(x) + i\sin(x))^n = (e^{ix})^n = e^{inx} = \cos(nx) + i\sin(nx).$$

(b) We know that raising a complex number to a power also raises its modulus to the same power. So, to be a solution to $x^n = 1$, we need $|x| = 1$. So, we have $x = e^{i\theta}$ for some angle θ. Then, by part (a), we get that $\cos(n\theta) + \sin(n\theta) = 1$. So, we have $n\theta = 2k\pi$ for some $k \in \mathbb{Z}$. Thus, the solutions are $\theta = 0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, \ldots, (n-1)\frac{2\pi}{n}$, and the solutions for x are $x = 1, x^{\frac{2\pi}{n}}, x^{2\frac{2\pi}{n}}, \ldots, x^{(n-1)\frac{2\pi}{n}}$.

2. Find all the solutions of the following equations in $\mathbb{C}$.
   (a) $z^2 + 6z + 12 - 4i = 0$
   (b) $(z^3 - 1)^2 = -1$

Solution:

(a) First, we try completing the square (using the quadratic formula is analogous). So, we have

$$z^2 + 6z + 12 - 4i = z^2 + 6z + 9 + 3 - 4i = (z+3)^2 + 3 - 4i.$$

So, our equation becomes

$$(z+3)^2 = 4i - 3.$$

With some work involving trigonometry, one can show that $1 + 2i$ is a square root of $4i - 3$. Then, the other root is $-1 - 2i$. So, one solution will be $z + 3 = 1 + 2i$, namely $z = -2 + 2i$, and the other will be $z = -4 - 2i$. In this process, we relied on the given fact that $1 + 2i$ is a square root of $4i - 3$. This may require some lengthy work. For this reason, sometimes it is best to consider an alternative approach, which is discussed here below.

We consider the form $z = a + ib$ with $a, b \in \mathbb{R}$. Substituting this into the equation we get,

$$(a + ib)^2 + 6(a + ib) + 12 - 4i = 0.$$

which is equivalent to the system of equations

$$a^2 - b^2 + 6a + 12 = 0$$
$$2ab + 6b - 4 = 0.$$

from the first equation we obtain

$$a = -3 \pm \sqrt{b^2 - 3}$$

and since a and b are real numbers then $|b| \geq \sqrt{3}$. We can write the second equation as

$$a = \frac{2}{b} - 3$$

If we substitute this in the above expression we get that

$$b^2 - 3 = \frac{4}{b^2} \implies b^4 - 3b^2 - 4 = (b^2 - 4)(b^2 + 1) = 0$$

Then we have $b = \pm 2$ since b is real. So the two solutions to the equations are $z_1 = -2 + 2i$ and $z_2 = -4 - 2i$.

Verification: substitute the solutions into the equation and compute.

(b) Use the change of variable $\tilde{z} = z^3 - 1$. Now the equation for $\tilde{z}$ takes the form

$$\tilde{z}^2 = -1$$

so $\tilde{z} = \pm i$. So $z^3 = 1 \pm i$. We write

$$1 + i = \sqrt{2} e^{i\frac{\pi}{4}}$$
$$1 - i = \sqrt{2} e^{-i\frac{\pi}{4}}$$

We use the polar representation of complex numbers and write $z = re^{i\theta}$ with $r > 0$ and $0 \leq \theta < 2\pi$. For $z^3 = \sqrt{2}e^{i\pi/4}$ we have

$$z^3 = (re^{i\theta})^3 = r^3 e^{i(3\theta)} = \sqrt{2} e^{i(2k\pi + \pi/4)}, \quad k = 0, 1, 2, \ldots$$

which has the solutions $r = \sqrt[6]{2}$ and $\theta = \frac{\pi}{12}$, $\theta = \frac{3\pi}{4}$ and $\theta = \frac{17\pi}{12}$. Similarly for $z^3 = \sqrt{2}e^{-i\pi/4}$ we have

$$z^3 = (re^{i\theta})^3 = r^3 e^{i(3\theta)} = \sqrt{2} e^{i(2k\pi - \pi/4)}, \quad k = 0, 1, 2, \ldots$$

which has the solutions $r = \sqrt[6]{2}$ and $\theta = \frac{23\pi}{12}$, $\theta = \frac{7\pi}{12}$ and $\theta = \frac{5\pi}{4}$. The final 6 solutions are

$$z_1 = \sqrt[6]{2} e^{i\frac{\pi}{12}}$$
$$z_2 = \sqrt[6]{2} e^{i\frac{3\pi}{4}}$$
$$z_3 = \sqrt[6]{2} e^{i\frac{17\pi}{12}}$$
$$z_4 = \sqrt[6]{2} e^{i\frac{23\pi}{12}}$$
$$z_5 = \sqrt[6]{2} e^{i\frac{7\pi}{12}}$$
$$z_6 = \sqrt[6]{2} e^{i\frac{5\pi}{4}}$$

Verification: substitute the solutions into the equation and compute.

3. (a) Let $\{a_n\}$ be a sequence and let $\{b_n\}$ be the sequence defined as $b_n := |a_n|$. Prove that $\{a_n\}$ is bounded if and only if so is $\{b_n\}$.

   (b) Prove that if $q < -1$, then the sequence $x_n = aq^n$, $a \in \mathbb{R}$, $a \neq 0$, is unbounded.

   (c) Provide an example of a sequence $\{a_n\}$ that is bounded above and such that $\{|a_n|\}$ is not bounded above.

Solution:

1. Let us prove that the boundedness of $\{a_n\}$ implies the boundedness of $\{b_n\}$. The sequence $\{a_n\}$ is bounded $\iff_{\text{by definition}}$ there exists real number $C_1, C_2 \in \mathbb{R}$ such that
$$C_1 \leq a_n \leq C_2, \quad \forall n \in \mathbb{N}.$$
Let $C := \max\{|C_1|, |C_2|\}$. Then,
$$-C \leq C_1 \leq a_n \leq C_2 \leq C, \quad \forall n \in \mathbb{N}, \tag{1}$$
by the properties of the aboslute values of real numbers. We can rewrite (1) as
$$|a_n| \leq C, \quad \forall n \in \mathbb{N}. \tag{2}$$
As $|a_n| \geq 0$, then $0 \leq |a_n| \leq C$, for all $n \in \mathbb{N}$ and b_n is bounded.
Let us prove that the boundedness of $\{b_n\}$ implies the boundedness of $\{a_n\}$. The sequence $\{b_n\}$ is bounded $\iff_{\text{by definition}}$ there exists a non-negative real number C' such that $0 \leq b_n \leq C'$, $\forall n \in \mathbb{N}$ (since $b_n \geq 0$) $\iff_{b_n = |a_n|}$ there exists a non-negative real number C' such that $-C' \leq a_n \leq C'$ $\forall n \in \mathbb{N} \iff_{\text{by definition}}$ (a_n) is bounded.

2. We consider the sequence $|x_n| = a|q|^n$, where now we know $|q| > 1$. Then, we saw in class that this sequence is not bounded (using Bernoulli's inequality). Then, by part (a) we are done.

3. The sequence $a_n = -n$ is a valid example.

4. Let $\{x_n\}$ be the recursive sequence defined as
$$\begin{cases} x_n = x_{n-1} + (-1)^n n^2 \\ x_0 = 0. \end{cases}$$
Prove that for all $m \in \mathbb{N}$
$$\begin{cases} x_{2m} = (2m+1)m \\ x_{2m+1} = -(2m+1)(m+1). \end{cases}$$
[*Hint*: use induction on m].
Is $\{x_n\}$ bounded from above? Is it bounded from below?

Solution:

We will prove this by induction. One can easily verify the formula for $m = 0$: indeed, $0 = x_0 = x_{2\cdot 0} = (2 \cdot 0 + 1) \cdot 0$.

Now suppose the claim holds for some natural number $m = k$; that is, we have $x_{2k} =$

$(2k+1)k$ and $x_{2k+1} = -(2k+1)(k+1)$. Then,

$$\begin{aligned} x_{2k+2} =& x_{2k+1} + (-1)^{2k+2}(2k+2)^2 \\ =& -(2k+1)(k+1) + (2k+2)^2 \\ =& (2k+3)(k+1) \end{aligned}$$

and

$$\begin{aligned} x_{2k+3} =& x_{2k+2} + (-1)^{2k+3}(2k+3)^2 \\ =& (2k+3)(k+1) - (2k+3)^2 \\ =& -(2m+3)(m+2) \end{aligned}$$

as required. From this we also see that the sequence is not bounded from below nor from above, since x_{2m} is monotonic increasing and unbounded and x_{2m+1} is monotonic decreasing and unbounded.

5. Check if the sequence starting from $n = 1$ defined as $a_n = \frac{\sin(\frac{1}{n})}{n}$ is monotone, and if it converges or diverges.

Solution: Consider the unit circle. Since the sin function is defined as the y-coordinate of a point on the unit circle, we have that, if $0 \le p \le q \le \frac{\pi}{2}$, we have $0 = \sin(0) \le \sin(p) \le \sin(q) \le \sin(\frac{\pi}{2}) = 1$. Since $0 \le \frac{1}{n} \le 1 \le \frac{\pi}{2}$ for every positive integer, we have that $0 \le \sin(\frac{1}{n}) \le \sin(1) \le \sin(\frac{\pi}{2}) = 1$ for every positive integer. So, the numerator is bounded between 0 and 1. Then, as the denominator is always greater than or equal than 1, we get $a_n \le \sin(\frac{1}{n})$. In particular, the sequence $\{a_n\}$ is bounded by 0 and 1.

For every positive integer n, we have $0 \le \frac{1}{n+1} \le \frac{1}{n} \le \frac{\pi}{2}$. So, we have $0 \le \sin(\frac{1}{n+1}) \le \sin(\frac{1}{n}) \le 1$. In particular, $\sin(\frac{1}{n})$ is decreasing. Dividing a positive decreasing sequence, such as $\sin(\frac{1}{n})$, by a positive increasing one, such as n, we obtain a decreasing sequence, so a_n is monotone decreasing (notice that here we need the sequence to be positive, as having negative signs may swap the order of inequalities). We conclude that a_n, being monotone and bounded, converges. Another way to prove the convergence is to observe that since $0 \le \sin(\frac{1}{n}) \le 1$ for all $n \ge 1$, then $0 \le a_n \le \frac{1}{n}$. So a_n converges to 0 by the Squeeze Theorem.

6. Check if the sequence
   (a) $a_n = \frac{n}{4n-1}$
   (b) $a_n = (-1)^n \frac{n^2+\pi}{n}$ starting from $n = 1$

   is monotone, and if it converges or diverges.

Solution:

(a) Since $a_{n+1} - a_n = -\frac{1}{(4n-1)(4n+3)} < 0$, the sequence a_n is monotone decreasing. It is bounded below by 0, hence it converges. Indeed, $a_n = \frac{1}{4-\frac{1}{n}}$, so for all $\varepsilon > 0$ we have $0 \le \frac{1}{4-\frac{1}{n}} - \frac{1}{4} \le \varepsilon$ for all $n \ge n_\varepsilon = \left[\frac{1}{16\varepsilon} + \frac{1}{4}\right] + 1$.

(b) It is not monotone as can be seen by the fact that it oscillates between being positive and negative. It also does not converge because the sequence is unbounded. This is seen by $|a_n| = \frac{n^2+\pi}{n} = n + \frac{\pi}{n}$.

7. Find the limit of the following sequences, if they exist:

   (a) $a_n = \dfrac{5n^2 - 3n + 2}{3n^2 + 7}$

   (b) $a_n = (-1)^n \dfrac{\sqrt[4]{n}}{\sqrt[3]{n}}$

   (c) $a_n = \frac{\sqrt{n}-n+n^2}{2n^2+n^{\frac{3}{2}}+n}$

   (d) $a_n = \sin(\frac{1}{n}) + \frac{n-2}{n\sqrt{2}+77}$

Solution:

(a)

$$\lim_{n\to\infty} \frac{5n^2 - 3n + 2}{3n^2 + 7} = \lim_{n\to\infty} \frac{5 - 3\frac{1}{n} + \frac{2}{n^2}}{3 + \frac{7}{n^2}} = \frac{5 - 3 \lim\limits_{n\to\infty} \frac{1}{n} + 2 \lim\limits_{n\to\infty} \frac{1}{n^2}}{3 + 7 \cdot \lim\limits_{n\to\infty} \frac{1}{n^2}} = \frac{5}{3}$$

(b) We have

$$\lim_{n\to\infty} \left| (-1)^n \frac{\sqrt[4]{n}}{\sqrt[3]{n}} \right| = \lim_{n\to\infty} n^{\left(\frac{1}{4}-\frac{1}{3}\right)} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{12}}} = 0.$$

Since $-|a_n| \le a_n \le |a_n|$ for all n, by the Squeeze Theorem we have

$$\lim_{n\to\infty} (-1)^n \frac{\sqrt[4]{n}}{\sqrt[3]{n}} = 0\ .$$

(c) Dividing numerator and denominator by n^2 we get

$$a_n = \frac{n^{-\frac{3}{2}} - n^{-1} + 1}{2 + n^{-\frac{1}{2}} + n^{-1}}$$

so, arguing as in part (a), the limit is $\frac{1}{2}$.

(d) As argued in problem 4, $\sin(\frac{1}{n})$ converges to 0. Furthermore, arguing as in part (a), $\frac{n-2}{n\sqrt{2}+77}$ converges to $\frac{1}{\sqrt{2}}$; both limits exist, so the sum converges to the sum of the limits, which is $\frac{1}{\sqrt{2}}$

8. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

   (a) If $\{a_n\}$ is bounded then $\{a_n\}$ is convergent.

   (b) If $\{a_n\}$ is bounded and $a_n \ge 0$, $\forall n \in \mathbb{N}$, then $\{a_n\}$ is convergent.

   (c) If $\{a_n\}$ is monotone and unbounded, then it is bounded from above.

   (d) If $\{a_n\}$ is monotone and unbounded, then it is bounded from below.

(e) If $\{a_n\}$ is bounded and monotone then $\{a_n\}$ is convergent.

(f) If $\{a_n\}$ is convergent, then there exists $\epsilon > 0$ such that $|a_n| \leq \epsilon$ for all $n \in \mathbb{N}$.

(g) Let $\{a_n\}$ be a sequence and let $\{b_n\}$ be the sequence defined as $b_n := |a_n|$. Then, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} b_n = 0$.

Solution:

(a) False, take the sequence $a_n = (-1)^n$.

(b) False, take the sequence $a_n = (-1)^n + 1$.

(c) False, take the sequence $a_n = n$.

(d) False, take the sequence $a_n = -n$.

(e) True. Suppose that a_n is monotone increasing. We prove that $\lim\limits_{n\to+\infty} a_n = \sup\limits_{n\in\mathbb{N}} a_n := a$.

Let $\varepsilon > 0$ be given. By the definition of supremum $\exists N \in \mathbb{N}$ s.t. $|a - a_N| = a - a_N < \varepsilon$. Let $n > N$, by monotonicity we have $|a - a_n| = a - a_n \leq a - a_N < \varepsilon$. Thus, $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a - a_n| < \varepsilon$ as required.

Now suppose that a_n is monotone decreasing. We prove that $\lim\limits_{n\to+\infty} a_n = \inf\limits_{n\in\mathbb{N}} a_n := a$.

Let $\varepsilon > 0$ be given. By the definition of infimum $\exists N \in \mathbb{N}$ s.t. $|a_N - a| = a_N - a < \varepsilon$. Let $n > N$, by monotonicity we have $|a_n - a| = a_n - a \leq a_N - a < \varepsilon$. Thus, $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a - a_n| < \varepsilon$ as required.

(f) True. Convergent sequences are bounded. If $a_n \to a$ then $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a_n - a| < \frac{1}{17} \Rightarrow |a_n| = |a_n - a + a| \leq |a_n - a| + |a| < \frac{1}{17} + |a|$. Thus, $\forall n > N \quad |a_n| < \frac{1}{17} + |a|$. It therefore follows that $|a_n| \leq \max\{|a_1|, \cdots, |a_{N-1}|, \frac{1}{17} + |a|\} \quad \forall n \in \mathbb{N}$. Hence, we may choose $\epsilon = \max\{|a_1|, \cdots, |a_{N-1}|, \frac{1}{17} + |a|\}$.

(g) True. 0 is the limit of $(a_n) \Longleftrightarrow_{\text{by definition}} \forall \epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}$ such that for $n \geq n_\epsilon$, $|a_n - 0| = |\frac{1}{n}| < \epsilon \Longleftrightarrow_{\text{since } |\frac{1}{n}| = b_n} \forall \epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}$ such that for $n \geq n_\epsilon$, $|b_n - 0| = |\frac{1}{n}| < \epsilon \Longleftrightarrow_{\text{by definition}} \lim_{n\to\infty} b_n = 0$.

9. Let $p > q$ be natural numbers. Show that if $P(x) = \sum_{i=0}^{p} c_i x^i$ is a polynomial with real coefficients of degree p (that is, $c_p \neq 0$), and $Q(x) = \sum_{j=0}^{q} b_j x^j$ is a polynomial with real coefficients of degree q (that is, $b_q \neq 0$), then the sequence (a_n) defined as

$$a_n := \frac{P(n)}{Q(n)} \quad \text{is unbounded.}$$

Solution:

By the contrapositive of exercise 7(c) above, it suffices to show that for any positive real number C, there exists a natural number $n_C \in \mathbb{N}$ such that $|a_{n_C}| \geq C$. Let us assume by contradiction that $\{a_n\}$ is bounded. Then, there exists positive real number Csuch that for all $n \in \mathbb{N}$ then $|a_{n_C}| < C$. Then,

$$0 \leq \left(\frac{1}{n}\right)^{p-q} |a_n| \leq \left(\frac{1}{n}\right)^{p-q} C.$$

As $p > q$, then $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{p-q} = 0$. Hence, by the Squeeze Theorem, the above 2 equations imply that $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{p-q} |a_n| = 0$. On the other hand,

$$\left(\frac{1}{n}\right)^{p-q} |a_n| = |\frac{P(n)}{n^{p-q}Q(n)}| = |\frac{P(n)}{Q_1(n)}|,$$

where the polynomial $Q_1(x)$ is defined as $Q_1(x) = x^{p-q}Q(x)$. Then the degree of $Q_1(x)$ is exactly p and the coefficient of leading term of Q_1 (that is the coefficient of the monomial of degree p in Q_1) is $b_q \neq 0$. Hence, the result that we discussed in class, tell us that since P, Q_1 have the same degree then

$$\lim_{n\to\infty} |\frac{P(n)}{Q_1(n)}| = |\frac{a_p}{b_q}| \neq 0,$$

since $a_p \neq 0$. But this is contradiction with what we proved before.

10. Find the limit of the following sequences, if they exist:
    (a) $a_n = \sqrt{2n^2+3} - \sqrt{(2n+1)(n+4)}$
    (b) $a_n = \sqrt{n}(\sqrt{n^3+2n} - \sqrt{n^3+4})$

Solution:

(a)

$$\lim_{n\to\infty} \sqrt{2n^2+3} - \sqrt{(2n+1)(n+4)} = \lim_{n\to\infty} \frac{2n^2+3-(2n+1)(n+4)}{\sqrt{2n^2+3}+\sqrt{(2n+1)(n+4)}}$$
$$= \lim_{n\to\infty} \frac{-9n-1}{\sqrt{2}n\left(\sqrt{1+\frac{3}{2n^2}}+\sqrt{1+\frac{9}{2n}+\frac{2}{n^2}}\right)} = \frac{-9}{2\sqrt{2}}.$$

(b)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n^3+2n} - \sqrt{n^3+4}) = \lim_{n\to\infty} \frac{\sqrt{n}(n^3+2n-n^3-4)}{\sqrt{n^3+2n}+\sqrt{n^3+4}}$$
$$= \lim_{n\to\infty} \frac{\sqrt{n}(2n-4)}{n\sqrt{n}\left(\sqrt{1+\frac{2}{n^2}}+\sqrt{1+\frac{4}{n^3}}\right)} = 1$$

11. Find the limit of the following sequences:
    (a) $a_n = \sin\left(\frac{1}{n}\right)$
    (b) $a_n = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

(c) $a_n = n \cdot \sin\left(\dfrac{2n+3}{n^3}\right)$

Hint: remember that for $0 < x < \pi/2$ we have the inequalities:

$$0 \le \sin(x) \le x \le \tan(x) \quad \Rightarrow \quad 1 \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)} \quad \Rightarrow \quad \cos(x) \le \frac{\sin(x)}{x} \le 1$$

$$\Rightarrow \quad \cos(x)^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1 \quad \Rightarrow \quad 1 - \sin(x)^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1$$

$$\Rightarrow \quad 1 - x^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1 \quad \Rightarrow \quad \sqrt{1-x^2} \le \frac{\sin(x)}{x} \le 1 \ .$$

Solution:

(a) We have

$$0 \le \sin\left(\frac{1}{n}\right) \le \frac{1}{n} \quad \overset{\text{Squeeze Theorem}}{\Longrightarrow} \quad \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0 \ .$$

(b) We have

$$\sqrt{1 - \frac{1}{n^2}} \le \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \le 1 \quad \overset{\text{Squeeze Theorem}}{\Longrightarrow} \quad \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$$

(c) We have

$$\sqrt{1 - \left(\frac{2n+3}{n^3}\right)^2} \le \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}} \le 1 \ .$$

Similarly to part (b), we first argue that

$$\lim_{n\to\infty} \sqrt{1 - \left(\frac{2n+3}{n^3}\right)^2} = 1 \ .$$

According to the Squeeze Theorem

$$\lim_{n\to\infty} \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}} = 1 \ .$$

So

$$\lim_{n\to\infty} \left(n \sin\left(\frac{2n+3}{n^3}\right)\right) = \lim_{n\to\infty} \left(\frac{2n+3}{n^2} \cdot \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}}\right)$$

$$= \left(\lim_{n\to\infty} \frac{2n+3}{n^2}\right) \cdot \left(\lim_{n\to\infty} \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}}\right) = 0 \cdot 1 = 0 \ .$$

Note that we can split the limits because both sequences $\left(\frac{2n+3}{n^2}\right)$ and $\left(\frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}}\right)$ converge.

12. Show that the sequence given by

$$a_1 = 2$$
$$a_n = \frac{1}{2}(a_{n-1} + 6)$$

is increasing and bounded above by 6. (*Hint: Use induction for both*)

Solution:

It is slightly easier to show first that the sequence is bounded and then that it is monotone, and we will indeed apply this strategy in the next exercises. Here, let's show first that it is monotone.

To show that the sequence is increasing we define the difference sequence as $d_n = a_{n+1} - a_n = 3 - a_n/2$. Showing $d_n \geq 0$ is equivalent to showing (a_n) is increasing. We show by induction: for $n = 1$, $d_1 = 2 > 0$. Now suppose that $d_n \geq 0$ meaning $a_n \leq 6$. For $n + 1$ we have:

$$d_{n+1} = 3 - \frac{a_{n+1}}{2} = 3 - \frac{3 + a_n/2}{2} = \frac{3}{2} - \frac{a_n}{4} \geq \frac{3}{2} - \frac{6}{4} = 0$$

So (a_n) is increasing. To show that it is bounded above by 6 we use induction again. For $n = 1$ we have that $a_1 = 2 < 6$. Suppose that $a_n < 6$, for $n + 1$ we have:

$$a_{n+1} = \frac{1}{2}(a_n + 6) \leq \frac{1}{2}(6 + 6) = 6.$$

13. Let $\{a_n\}$ be a sequence. Specify if the following statements are true or false. If you believe that the statement is true, you should give a proof, otherwise, provide a counterexample to the statement.

(a) If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$

then $\{a_n\}$ converges.

(b) If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$

then $\{a_n\}$ diverges.

Solution:

(a) False, take $a_n = n$.

(b) False, take for example $a_n = 1/n$. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$

but $\lim_{n\to\infty} \frac{1}{n} = 0$.

14. Consider two sequences of real numbers a_n and b_n. Assume that $0 < a_n < 3$ and $-4 < b_n < 0$ for every n. Which of the following claims is true? (Only one choice is correct)

(a) The sequence $\frac{1}{a_n}$ is bounded.

(b) The sequence $a_n b_n$ is bounded below by -4.

(c) The sequence $a_n + b_n$ has to be negative.

(d) The sequence $a_n b_n$ is bounded below by -12.

(e) The sequence $\frac{a_n}{b_n}$ is bounded.

Solution: The right answer is (d). Indeed, for every n, we have

$$-12 = -4 \cdot 3 < -4 \cdot a_n < b_n \cdot a_n,$$

where the first inequality comes from multiplying $a_n < 3$ both sides by -4, and the second inequality comes from multiplying $-4 < b_n$ both sides by the positive number a_n.

A counterexample to (a) is given by $a_n = \frac{1}{n}$.

A counterexample to (b) is given by the constant sequences $a_n = 2$ and $b_n = -3$.

A counterexample to (c) is given by the constant sequences $a_n = 2$ and $b_n = -1$.

A counterexample to (e) is given by $a_n = 1$ (constant) and $b_n = -\frac{1}{n}$.

15. Let z and w be two complex numbers. Which of the following statements is true? (Only one choice is correct)

(a) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{w}$

(b) $|z| = z \cdot \overline{z}$

(c) $\mathrm{Im}(z+w) = \mathrm{Re}(i(z+w))$

(d) $\mathrm{Re}(z+w) = \mathrm{Im}(i(z+w))$

(e) $i\mathrm{Re}(z+w) = \mathrm{Im}(z+w)$

Solution: Let $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$.

The right answer is (d). Indeed, $z+w = (a+c)+i(b+d)$, so $\mathrm{Re}(z+w) = a+c$. Similarly, we have $i(z+w) = -(b+d) + i(a+c)$, and $\mathrm{Im}(i(z+w)) = a+c$.

Answer (a) is incorrect, since we need to conjugate the denominator as well to obtain a right formula. A counterexample to the formula given in (a) is given by $z = i$ and $w = i$.

Answer (b) is incorrect, since we need to take the square root of $z \cdot \overline{z}$ to get the absolute value of z.

The above computation for (d) shows that (c) is wrong, as the formula also needs a negative sign to hold true.

Finally, answer (e) is incorrect, since real and imaginary parts are real numbers.

16. Prove the following properties of the binomial coefficients:

(a) Symmetry: $\binom{n}{k} = \binom{n}{n-k}$;

(b) Binomial formula: Assuming the recurrence formula that you find in (c) below, prove that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. [*Hint*: use induction on n. You may use the result from (c).]

(c)* Recurrence: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Deduce from this that $\binom{n}{k} \in \mathbb{N}$; [*Hint*: use induction on n.]

Solution:

(a) $\binom{n}{k} = \frac{n!}{k!(n-k!)} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

(b) Prove by induction. The formula for $n = 0$ can be easily verified (remember $0! = 1$ and $\binom{0}{0} = 1$). Now suppose it holds for some natural number m. Hence,

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}$$

Now show that this implies that the formula holds for $m+1$.

$$\begin{aligned}
&(x+y)^{m+1} \\
=&(x+y)^m(x+y) \\
=&\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}(x+y) \\
=&\sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} \\
=&x^{m+1} + \sum_{k=0}^{m-1} \binom{m}{k} x^{k+1} y^{m-k} + y^{m+1} + \sum_{k=1}^{m} \binom{m}{k} x^k y^{m+1-k} \\
=&x^{m+1} + \sum_{k=1}^{m} \binom{m}{k-1} x^k y^{m-k+1} + \sum_{k=1}^{m} \binom{m}{k} x^k y^{m+1-k} + y^{m+1} \\
=&x^{m+1} + \sum_{k=1}^{m} \left(\binom{m}{k} + \binom{m}{k-1} \right) x^k y^{m+1-k} + y^{m+1} \\
=&x^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} x^k y^{m+1-k} + y^{m+1} \\
=&\sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m+1-k}
\end{aligned}$$

as required.

(c) Follows from direct computation:

$$\begin{aligned}
&\binom{n-1}{k-1} + \binom{n-1}{k} \\
=&\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
=&\frac{(n-1)!k!(n-k-1)! + (n-1)!(k-1)!(n-k)!}{k!(n-k)!(k-1)!(n-k-1)!} \\
=&\frac{1}{k!(n-k)!}\frac{(n-1)!k!(n-k-1)! + (n-1)!(k-1)!(n-k)!}{(k-1)!(n-k-1)!} \\
=&\frac{1}{k!(n-k)!}\frac{(n-1)!k + (n-1)!(n-k)}{1} \\
=&\frac{n(n-1)!}{k!(n-k)!} \\
=&\binom{n}{k}
\end{aligned}$$

as required. Now we can show that they are all contained in $\mathbb{N}$ by induction on n. One can verify that it holds for $n = 0$. Now suppose it holds for some $m - 1$. Then $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$ must also be a natural number since it is a sum of two natural numbers.