Analysis I (English)
T Mountford Fall Semester 2022–2023

Analysis 1 - Exercise Set 3

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Let $[x]$ denote the integral part of a number $x \in \mathbb{R}$.
   Prove that, for every $x \in \mathbb{R}$, $[x] = -[-x]$.

Solution:
Recall the definition of integral part:

$$[x] = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x \leq 0. \end{cases}$$

The question is symmetric between x and $-x$ so we can take $x \geq 0$ and since $x = 0$ is immediate as $[0] = 0$ we can suppose that $x > 0$. In this case $x = [x] + a$, where $0 \leq a < 1$.Again we can suppose that $a > 0$ since if $a = 0$, then $x = \lfloor x \rfloor = \lceil x \rceil = [x]$ and similarly $-x = \lfloor -x \rfloor = \lceil -x \rceil = [-x]$.

So $-x = -[x] - a$ for $0 < a < 1$. But in this case $[-x] = -[x]$ which we wanted to prove.

2. Let $a, b \in \mathbb{R}$. Prove that $||a| - |b|| \leq |a - b|$ and $||a| - |b|| \leq |a + b|$

Solution: First, by the triangle inequality, we have

$$|a| = |a - b + b| \leq |a - b| + |b|. \tag{1}$$

Similarly, we have

$$|b| = |b - a + a| \leq |a - b| + |a|, \tag{2}$$

where we also used $|a - b| = |b - a|$. Rearranging (1), we get

$$|a| - |b| \leq |a - b|, \tag{3}$$

while rearranging (2), we get

$$|b| - |a| \leq |a - b|. \tag{4}$$

By definition of absolute value, one among $||a| - |b|| = |a| - |b|$ and $||a| - |b|| = |b| - |a|$ holds true. Thus, by (3) and (4), we get $||a| - |b|| \leq |a - b|$.

For the second part, we use the fact that $|-b| = |b|$ and we apply the first part to the real numbers a and $-b$. Indeed, we get

$$||a| - |b|| = ||a| - |-b|| \leq |a - (-b)| = |a + b|.$$

3. Compute $\sup S$ and $\inf S$ where $S \subseteq \mathbb{R}$ is defined as

(a) $S := \bigcup_{n=1}^{\infty} [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$. Does S admit maximum and/or minimum?

(b) $S := \bigcap_{n=1}^{\infty} (-1 - \frac{1}{n}, 1 + \frac{1}{n})$. Does S admit maximum and/or minimum?

Solution:

(a) Note $S = (-1, 1)$. Thus, $\inf S = -1$ and $\sup S = 1$. They are not maxima or minima.

(b) Note $S = [-1, 1]$. Thus, $\inf S = -1$ and $\sup S = 1$. They are maxima or minima.

4. Compute $\min S$ where $S \subseteq \mathbb{N}$ is defined as

   (a) $S := \{n \in \mathbb{N} : \sqrt{n} > 17\}$

   (b) $S := \{n \in \mathbb{N} : \sum_{i=1}^{n} i \geq 17\}$

   (c) $S := \{n \in \mathbb{N} : \sum_{i=1}^{n} 2^{-i} > 1.7\}$

Solution:

(a) $n \in S \Leftrightarrow n > 17^2$. Thus, the minimum of S is $17^2 + 1$.

(b) Note $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. First find $x \in \mathbb{R}_+$ s.t. $\frac{x(x+1)}{2} = 17$ which gives $x = \frac{-1+\sqrt{137}}{2}$ from the quadratic formula. Now note $11^2 < 137 < 12^2$ hence $x \in [\frac{-1+11}{2}, \frac{-1+12}{2}] = [5, 5.5]$. Thus, the minimum of S is $\lceil x \rceil = 6$.

(c) Note $\sum_{i=1}^{n} 2^{-i} = \frac{1}{2} \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1 - (\frac{1}{2})^n < 1.7$. Thus, the set is empty and the minimum is not defined.

.

5. Compute $\max S$ where $S \subseteq \mathbb{Z}$ is defined as

   (a) $S = \{n \in \mathbb{Z} \mid n \neq 0 \text{ and } n + \frac{20}{n} < 9\}$

   (b) $S = \{n \in \mathbb{Z} \mid (\sqrt{3})^n \leq 10^{17}\}$.

   (c) $S = \{n \in \mathbb{Z} \mid \alpha^n \leq C\}$ where $\alpha > 1$ and $C > 1$ are constants. [You must discuss how $\max S$ varies, when α and C vary.]

Solution:

(a) If $n > 0$, $n + \frac{20}{n} < 9 \Leftrightarrow 0 > n^2 - 9n + 20 = (n-4)(n-5)$ from which we see that if we are restricted to natural numbers this can never be satisfied, as the solution of $0 > x^2 - 9x + 20 = (x-4)(x-5)$ is $4 < x < 5$. So, we may assume $n < 0$. Then, we see that $n + \frac{20}{n}$ is always negative, as $n < 0$. So, the inequality is always satisfied if if $n < 0$. Thus, S coincides with the set of negative integers. So, its maximum is -1.

(b) See the solution for (c) but substitute $C = 10^{17}$ and $\alpha = \sqrt{3}$.

(c) We seek the largest $n \in \mathbb{Z}$ s.t. $\alpha^n \leq C$. The natural logarithm is an increasing function. Thus, taking logarithms both sides of the inequality preserves the inequality, and we get $n \ln(\alpha) = \ln(\alpha^n) \leq \ln(C)$. Thus, the maximum of S is $\lfloor \frac{\ln(C)}{\ln(\alpha)} \rfloor$.

6. For the following complex numbers z compute the real and imaginary part, the complex conjugate $\bar{z}$, the absolute value $|z|$, the argument (also called phase) $\arg(z)$ and the inverse z^{-1}:

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2} i; \quad z = 16i; \quad z = 2 + 3i - 3e^{i\frac{\pi}{2}}; \quad z = e^{-5\pi i} + i.$$

Solution:

(a) $\mathrm{Re}z = \frac{1}{2}$; $\mathrm{Im}z = \frac{\sqrt{3}}{2}$; $\bar{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$; $\arg z = \frac{\pi}{3}$; $|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$;

(b) $\mathrm{Re}z = 0$; $\mathrm{Im}z = 16$; $\bar{z} = -16i$; $\arg z = \frac{\pi}{2}$; $|z| = 16$;

(c) $e^{i\frac{\pi}{2}} = i$, then $z = 2$; $\mathrm{Re}z = 2$; $\mathrm{Im}z = 0$; $\bar{z} = 2$; $\arg z = 0$; $|z| = 2$;

(d) $e^{-5\pi i} = -1$, since fot any $n \in \mathbb{Z}$, $e^{(t+2\pi)i} = e^{ti}$ and $e^{-\pi i} = -1$; then $z = -1 + i$; $\mathrm{Re}z = -1$; $\mathrm{Im}z = 1$; $\bar{z} = -1 - i$; $\arg z = \frac{3\pi}{4}$; $|z| = \sqrt{2}$.

7. Write the following complex numbers in the form $x + iy$.

(a) i^{17}

(b) $\frac{4-i}{3-2i}$

(c) $2i(i-1) + \left(\overline{\sqrt{3}+i}\right)^3 + (1+i)\overline{(1+i)}$

Solution:

(a) $i^{17} = i \cdot i^{16} = i \cdot (i^4)^4 = i \cdot (1)^4 = i$

(b)

$$\frac{4-i}{3-2i} = \frac{4-i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{12+2-3i+8i}{9+4} = \frac{14+5i}{13} = \frac{14}{13} + i\frac{5}{13}$$

(c)

$$\begin{aligned} 2i(i-1) &= 2(-1-i) = -2-2i, \\ \left(\overline{\sqrt{3}+i}\right)^3 &= \left(\sqrt{3}-i\right)^3 = \left(\sqrt{3}-i\right)^2 \left(\sqrt{3}-i\right) = \left(3-1-2i\sqrt{3}\right)\left(\sqrt{3}-i\right) \\ &= \left(2-2i\sqrt{3}\right)\left(\sqrt{3}-i\right) = 2\sqrt{3}-2i-6i-2\sqrt{3} = -8i, \\ (1+i)\overline{(1+i)} &= |1+i|^2 = 2. \end{aligned}$$

So $2i(i-1) + \left(\overline{\sqrt{3}+i}\right)^3 + (1+i)\overline{(1+i)} = -2-2i-8i+2 = -10i$.

8. Compute

   (a) $(1+i\sqrt{3})^{1980}$

   (b) $(1+i\sqrt{3})^{1988}$

> **Solution:**
>
> (a) We have
> $$(1+\sqrt{3}i)^{1980} = 2^{1980} e^{i\frac{1980\pi}{3}} = 2^{1980}$$
>
> (b) We have
> $$(1+\sqrt{3}i)^{1988} = 2^{1988} e^{i\frac{1988\pi}{3}} = 2^{1988}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) =$$
> $$= 2^{1988}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2^{1987}(-1+\sqrt{3}i).$$

9. Find all the solution of the following equations in $\mathbb{C}$. [The unknown is $z = x+iy$, or, if you prefer you could use polar form.]

   (a) $z^2 = i$

   (b) $z^5 = 1$.

   (c) $z^2 = -3+4i$.

> **Solution:**
>
> (a) We can either use Euler's formula or we can solve it directly as described here. We are searching for $x+iy$ such that $(x+iy)^2 = i$ meaning $(x^2-y^2)+i(2xy) = i$. Clearly $x^2-y^2 = 0$ and $2xy = 1$. From the first equation we deduce that $x = \pm y$.
> $$x = y \Longrightarrow x\cdot(x) = \frac{1}{2} \Longrightarrow x = \pm\frac{\sqrt{2}}{2}$$
> $$x = -y \Longrightarrow x\cdot(-x) = \frac{1}{2} \Longrightarrow x^2 = -\frac{1}{2} \quad \text{(not valid since } x\in\mathbb{R}\text{)}.$$
> So the roots of i are $\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$.
>
> To check the solution, one can compute $\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^2 - i$ and $\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)^2 - i$.
>
> (b) Write z using Euler's formula: $z = |z|e^{i\theta}$ where θ is the phase of z. Hence, $z^5 = |z|^5 e^{i5\theta} = e^{i2\pi k} \Rightarrow |z| = 1$ and $\theta = \frac{2\pi}{5}k$ for $k = 0,1,2,3,4$.
>
> (c) Write z using Euler's formula: $z = |z|e^{i\theta}$. Note $|-3+4i| = 5$ thus $|z|^2 = 5 \Rightarrow |z| = \sqrt{5}$. Note $2\theta = \arg(-3+4i) = \arctan(-\frac{4}{3}) + \pi + 2\pi k = \varphi + 2\pi k$. Thus, $\theta = \frac{\varphi}{2} + \pi k$ for $k = 0,1$.

10. In the context of complex numbers, state if the following statements are true or false.

    (a) There exists a $n\in\mathbb{N}$ such that $(1+i\sqrt{3})^n$ is pure imaginary.

(b) There exists a positive $n \in \mathbb{N}$ such that $(1 - i\sqrt{3})^n$ is real.

Solution:

(a) False. We have

$$(1+\sqrt{3}i)^n = 2^n e^{i\frac{n\pi}{3}} = 2^n \left(\cos(n\frac{\pi}{3}) + i\sin(n\frac{\pi}{3})\right)$$

For the complex number to be pure imaginary, we require $\cos(n\frac{\pi}{3}) = 0$ which means $n = \frac{3}{2} + 3k$ for some $k \in \mathbb{Z}$. This condition cannot be satisfied if $n \in \mathbb{N}^*$.

(b) True. Similar to the previous part we have

$$(1-\sqrt{3}i)^n = 2^n e^{-i\frac{n\pi}{3}} = 2^n \left(\cos(n\frac{\pi}{3}) - i\sin(n\frac{\pi}{3})\right)$$

it is sufficient to find some n such that $\sin(n\frac{\pi}{3}) = 0$. Take for example $n = 3$.

11. Show that for all $\theta \in \mathbb{R}$ and for all $n \in \mathbb{N}$

$$(\cos(\theta) + i\sin(\theta))^n = (\cos(n\theta) + i\sin(n\theta)).$$

Solution: By definition of polar representation, we have $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Then, by the property of the exponents, we have

$$(\cos(\theta) + i\sin(\theta))^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

12. Prove that for all $z_1, z_2 \in \mathbb{C}$,

(a) $z_1 = 0$ if and only if $|z_1| = 0$.

(b) $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{i(\alpha_1-\alpha_2)}$, where $z_2 \neq 0$ and

$$z_1 = |z_1|e^{i\alpha_1}, \quad z_2 = |z_2|e^{i\alpha_2}, \quad \alpha_1,\ \alpha_2 \in \mathbb{R}$$

are the polar forms of the z_i using Euler's formula.

(c) $|\frac{z_1}{|z_1|}| = 1$.

Solution:

(a) One direction is trivial. The other follows from if $z = x + iy$, then $0 = |z| = \sqrt{x^2+y^2} \geq |x| \Rightarrow x = 0$ and similar for y.

(b) $\frac{z_1}{z_2} = \frac{|z_1|e^{i\alpha_1}}{|z_2|e^{i\alpha_2}} = \frac{|z_1|}{|z_2|}e^{i(\alpha_1-\alpha_2)}$.

(c) Say $z_1 = a + bi \quad \Longrightarrow \quad |z_1| = \sqrt{a^2+b^2} \quad \Longrightarrow \quad \frac{z_1}{|z_1|} = \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}i$
$\Longrightarrow |\frac{z_1}{|z_1|}| = \sqrt{(\frac{a}{\sqrt{a^2+b^2}})^2 + (\frac{b}{\sqrt{a^2+b^2}})^2} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1.$

13. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $\mathrm{Im}(z(2-i)) = 1$ is

   (a) A point.
   (b) A line.
   (c) A circle.
   (d) Empty.

> **Solution:** (b) is correct. Writing $z = x+iy$, we get that $(x+iy)(2-i) = (2x+y)+i(2y-x)$ so the equation becomes
> $$2y - x = 1$$
> whose solutions are the points on the line.

14. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $\bar{z} = i(z-1)$ is

   (a) A point.
   (b) A line.
   (c) A circle.
   (d) Empty.

> **Solution:** (d) is correct. Write $z = x + iy$, with $x, y \in \mathbb{R}$ then the equation becomes,
> $$x - iy = i(x + iy - 1) \Longrightarrow x - iy = ix - y - i$$
> Meaning $x = -y$ and $-y = x - 1$ which has no solutions; we conlude that the equation has no solution.

15. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $z^2 \cdot \bar{z} = z$ is

   (a) A point.
   (b) A circle.
   (c) A point and a circle.
   (d) A disk.

> **Solution:** (c) is correct. We can write the equation as $z \cdot ((z\bar{z}) - 1) = 0$. This means that one solution is $z = 0$ which is one point. Also since $z\bar{z} = |z|^2$ another set of solution is $|z|^2 = 1$ which are all points of the circle of radius 1 centered at the origin.

16. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $|z + 3i| = 3|z|$ is

   (a) A point.
   (b) A line.
   (c) A circle.
   (d) Empty.

Solution:

(c) is correct. We square both terms and write $z = x + iy$ and we obtain

$$|z + 3i|^2 = |x + i(y + 3)|^2 = x^2 + (y + 3)^2, \qquad (3|z|)^2 = 9(x^2 + y^2)$$

So the equation turns into

$$x^2 + (y + 3)^2 = 9(x^2 + y^2) \Longleftrightarrow x^2 + y^2 - \frac{3}{4}y = \frac{9}{8} \Longleftrightarrow x^2 + \left(y - \frac{3}{8}\right)^2 = \left(\frac{9}{8}\right)^2.$$

Then the solution are all the points of the circle of radius 9/8 centered at (0, 3/8).

17. Given the function $f\colon \mathbb{C} \to \mathbb{C}$ defined as $f(z) = \frac{1+iz}{iz+i}$

   (a) find the domain of the function f. That is, determine the set $\mathrm{Dom}(f) \subseteq \mathbb{C}$ such that $z \in \mathrm{Dom}(f)$ if and only if $f(z)$ is defined;

   (b) find all complex numbers z such that $f(z) = z$;

   (c) find the preimages of $3 + i$.

Solution:

(a) Since f is a rational function (i.e., ratio of two polynomials), we need to determine when the denominator is not 0. So, $z \in \mathrm{Dom}(f)$ if and only if $iz + i \neq 0$. Dividing by i, we get $z + 1 \neq 0$. So, we conclude that $\mathrm{Dom}(f) = \mathbb{C} \setminus \{-1\}$.

(b) You have to solve the equation $f(z) = z$, so

$$\frac{1 + iz}{iz + i} = z$$

which turns out to be

$$z^2 = -i, \quad z \neq -1$$

the two solutions are $z = \pm\frac{1}{\sqrt{2}}(1 - i)$. Verification: substitute the solutions into the equation and compute. For example:

$$\frac{1 + i\frac{1-i}{\sqrt{2}}}{i\frac{1-i}{\sqrt{2}} + i} - \frac{1 - i}{\sqrt{2}} = \frac{\sqrt{2} + i + 1}{i + 1 + \sqrt{2}i} - \frac{1 - i}{\sqrt{2}} = \frac{2 + \sqrt{2}i + \sqrt{2} - i - 1 - \sqrt{2}i - 1 + i + -\sqrt{2}}{\sqrt{2}(i + 1 + \sqrt{2}i)} = 0.$$

(c) You have to solve

$$\frac{1 + iz}{iz + i} = 3 + i,$$

which is equivalent to

$$1 + iz = (3i - 1)z + 3i - 1, \quad z \neq -1.$$

This is linear in z, and the unique solution is $-\frac{8}{5} - \frac{1}{5}i$. Verification: compute $f(-\frac{8}{5} - \frac{1}{5}i)$.